

NOTIONS OF DOUBLE FOR LIE ALGEBROIDS *

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1 INTRODUCTION

The notion of double introduced by Drinfel'd [5] can be justified by philosophical considerations which lie at the heart of quantum theory. More concretely, the quantum double of a Hopf algebra and the classical double of a Lie bialgebra enable the complicated relations inherent in the original structures to be expressed in simpler terms on a larger object.

In this paper we are concerned with a specific aspect of the classical double of a Lie bialgebra: that a pair of Lie algebra structures on a vector space \mathfrak{g} and its dual \mathfrak{g}^* form a Lie bialgebra if and only if $\mathfrak{g} \oplus \mathfrak{g}^*$ admits a Lie algebra structure making $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ a Manin triple. The structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ may also be viewed as providing a concise formulation of the twisted derivation equations

$$\begin{aligned}\rho_X([Y_1, Y_2]) &= [\rho_X(Y_1), Y_2] + [Y_1, \rho_X(Y_2)] + \rho_{\sigma_{Y_2}(X)}(Y_1) - \rho_{\sigma_{Y_1}(X)}(Y_2), \\ \sigma_Y([X_1, X_2]) &= [\sigma_Y(X_1), X_2] + [X_1, \sigma_Y(X_2)] + \sigma_{\rho_{X_2}(Y)}(X_1) - \sigma_{\rho_{X_1}(Y)}(X_2),\end{aligned}$$

where $X_1, X_2 \in \mathfrak{g}$, $Y_1, Y_2 \in \mathfrak{g}^*$. Here ρ and σ are the two coadjoint representations, ρ of \mathfrak{g} on \mathfrak{g}^* and σ of \mathfrak{g}^* on \mathfrak{g} . The equations hold if and only if $(\mathfrak{g}, \mathfrak{g}^*)$, or equivalently $(\mathfrak{g}^*, \mathfrak{g})$, is a Lie bialgebra.

The Manin triple characterization of Lie bialgebras has been extended in two distinct directions. Firstly, several authors have independently extended the Manin triple theorem to pairs of Lie algebras not in duality. The notion of a pair of Lie algebras $\mathfrak{g}, \mathfrak{h}$ such that the vector space direct sum $\mathfrak{g} \oplus \mathfrak{h}$ has a Lie algebra structure for which \mathfrak{g} and \mathfrak{h} are Lie subalgebras was introduced in this context by Kosmann–Schwarzbach and Magri [13] under the name *extension bicroisée* or *twisted extension*, by Lu and Weinstein [18] as *double Lie algebra*, and by Majid [28], with the term *matched pair* which we use. (In fact, forms of the concept had been found much earlier; see [37], [2] for references.) A matched pair structure on a pair of Lie algebras $\mathfrak{g}, \mathfrak{h}$ is equivalent to a pair of representations, of \mathfrak{g} on \mathfrak{h} and of \mathfrak{h} on \mathfrak{g} , which satisfy the twisted derivation equations above.

In the case of Lie bialgebras, the notion of Poisson Lie group is a natural global concept [4]. It was shown by Lu and Weinstein [17] that a further step is possible: every Poisson Lie group may be integrated in a suitable sense to a double symplectic groupoid. The appearance of groupoids here may be understood as an instance of integrability for Poisson manifolds [10], [35], [3]: the symplectic double groupoid S of [17] corresponding to a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ incorporates two (symplectic) groupoid structures on S which realize the Poisson structures on G and G^* , the Poisson Lie groups which correspond to \mathfrak{g} and \mathfrak{g}^* .

Under suitable conditions this symplectic double groupoid is defined by two global actions, of G on G^* and of G^* on G , which integrate the infinitesimal dressing transformations; this gives (G, G^*) the structure of a matched pair of Lie groups as defined by [18], [28]. More generally, both [18] and [28] gave mild conditions for the global integrability of a matched pair of Lie algebras to a matched pair of Lie groups, with [28] using mainly the actions and [18] mainly the structure of the double.

In fact groupoid structures are present in any matched pair of Lie groups (G, H) : the action of H on G defines a groupoid structure on $S = G \times H$ with base G , and the action of G on H defines a groupoid structure on S with base H . These constitute a double groupoid structure on S , in the categorical sense of Ehresmann; that is, each groupoid structure on S makes it a groupoid object in the category of all groupoids. In this case there is the further property that elements of S , regarded as squares, are determined by any two adjacent sides.

It was shown in [20, §2] that any double groupoid with this property — there called *vacancy* — determines a matched pair structure on its side groupoids. One consequence of [17] is that a matched pair of Lie algebras which is not globally integrable to a matched pair of Lie groups may nonetheless integrate to a double groupoid.

The notion of matched pair has been extended to Lie groupoids [20, §2] and Lie algebroids [29]; Lu [16] demonstrated that any Poisson Lie group action gives rise to a matched pair of Lie algebroids which underlies Drinfel'd's classification of Poisson homogeneous spaces [6].

The second direction in which the Manin triple theorem may be extended is that of Poisson manifolds which are not groups. A Poisson structure on a general manifold P gives rise to a Lie algebroid structure on T^*P which, together with the standard tangent bundle structure on TP , forms a Lie bialgebroid in the sense of the author and Ping Xu [26]. In general a Lie bialgebroid (A, A^*) is a Lie algebroid A together with a Lie algebroid structure on A^* such that (as [12] showed) the coboundary induced by each is a derivation of the Schouten bracket of the other; thus Lie bialgebroids are strong differential Gerstenhaber algebras for which the underlying module structure comes from a vector bundle [39]. Lie bialgebroids arose in [26] as the infinitesimal invariants of the Poisson groupoids of Weinstein [36] (see also [38]); they have also arisen from the Poisson–Nijenhuis structures of Kosmann–Schwarzbach and Magri [14] and from the work on dynamical CYBE by Etingof and Varchenko [7]. In the latter case specialization of the algebraic structure leads to quite intricate systems of pde.

In the case of general Lie bialgebroids there are no representations corresponding to the coadjoint representations ρ and σ above. One can see immediately that a straightforward extension of the Drinfel'd double formula to Lie bialgebroids can be made only when the bialgebroid is in fact a bialgebra.

Recently Liu, Weinstein and Xu [15] have given an intricate and highly nonobvious extension of the Manin triple structure on the double of a Lie bialgebra to a bracket on $\Gamma(A \oplus A^*)$, for an arbitrary Lie bialgebroid (A, A^*) . The resulting structure is not usually a Lie algebroid, but its properties are abstracted by [15] into the notion of Courant algebroid; [15] shows that the Courant algebroid of a Lie bialgebroid is characterized by the existence of two complementary integrable isotropic subbundles. Thus the main criterion for a double is met.

Very recently, Roytenberg [32] has given a treatment of concepts of double in terms of super geometry.

In this paper we define a very general notion of “double Lie algebroid” which includes both the double of a matched pair of Lie algebroids, and the double of a Lie bialgebroid — in the bialgebra case, both reduce to the standard Manin triple theorem. Notice that these two generalizations of Lie bialgebras do not overlap except in the actual bialgebra case. At the same time, this notion of double incorporates such objects as the double (or iterated) tangent bundle $T^2M = T(TM)$ of a smooth manifold M , and other iterated tangent and cotangent structures; these, together with the canonical isomorphisms between them, are basic to some approaches to mechanics [34]. The word “double” here carries several meanings: it refers to the classical Drinfel'd double, as derived from the matched pair structure of a Lie bialgebra, and as derived from the duality between the Poisson structures in a Lie bialgebra, and it also refers to the categorical, or iterated, concept of double. The results of this paper in particular provide global objects corresponding to the double in all the cases discussed above.

A double Lie algebroid is first of all a double vector bundle as in Figure 1(a); that is, \mathcal{A} has two vector bundle structures, on bases A^V and A^H , each of which is itself a vector bundle on base M , such that for each structure on \mathcal{A} , the structure maps (projection, addition,

$$\begin{array}{ccc}
\mathcal{A} & \longrightarrow & A^V \\
\downarrow & & \downarrow \\
A^H & \longrightarrow & M
\end{array}
\quad
\begin{array}{ccc}
TA & \longrightarrow & TM \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}
\quad
\begin{array}{ccc}
T^*A & \longrightarrow & A^* \\
\downarrow & & \downarrow \\
A & \longrightarrow & M
\end{array}$$

(a) (b) (c)

Figure 1:

scalar multiplication) are vector bundle morphisms with respect to the other structure (see [30] or [20, §1]). Two cases to keep in mind are the tangent prolongation of an ordinary vector bundle as in Figure 1(b) (see [1] for a classical treatment), and the cotangent double vector bundle as in Figure 1(c) (see [26]).

Now suppose that all four sides of Figure 1(a) have Lie algebroid structures. The problem is to define compatibility between the bracket structures on \mathcal{A} , bearing in mind that the brackets are on different modules of sections. The key is the duality for double vector bundles introduced by Pradines [31]; see §2. Although we do not know how to interpret our compatibility condition as a direct condition of the form ‘either bracket structure on (sections of) \mathcal{A} is a morphism with respect to the other’, we show in Theorem 4.5 that this concept of double Lie algebroid includes the infinitesimal invariants of double Lie groupoids (and the \mathcal{LA} -groupoids of [20, §4]), which *are* defined by such conditions. The results of §4 were announced in [21], which also provides an overview of the background to the present paper.

In §5 we extend to abstract Lie bialgebroids the Manin triple characterization of Lie bialgebras. A Lie algebroid structure on a vector bundle A induces a Poisson structure on its dual A^* and this in turn induces a Lie algebroid structure on $T^*A^* \longrightarrow A^*$. If A^* also has a Lie algebroid structure, *a priori* unrelated, then $T^*A \longrightarrow A$ likewise has a Lie algebroid structure. Using the canonical isomorphism $R: T^*A^* \longrightarrow T^*A$ which interchanges A and A^* [26], these equip the cotangent double vector bundle of Figure 1(c) with four Lie algebroid structures. We prove in Theorem 5.2 that these constitute a double Lie algebroid if and only if (A, A^*) is a Lie bialgebroid. In the case of Lie bialgebras, 5.2 is equivalent to the Manin triple characterization, but in the general case it is necessary to use the side structures, with bases A and A^* , rather than a structure over M . Compared with the general Manin triple theorem of [15], the notion of double Lie algebroid involves a pair of (relatively) simple bracket structures, the relationship between which embodies the data, rather than the single but more complicated structure of a Courant algebroid. The results of §5 have been summarized in [21].

In §6 we show that matched pair structures on a pair of Lie algebroids A, B with base M correspond to double Lie algebroid structures on $A \times_M B$. Here $A \times_M B$ denotes the double vector bundle formed by the two pullbacks $q_A^!B$ and $q_B^!A$ across the projections $q_A: A \longrightarrow M$ and $q_B: B \longrightarrow M$; we call such double vector bundles *vacant*. The situation in §6 is that in addition to the Lie algebroid structures on A and B over M , we have Lie algebroid structures

on the two pullbacks $q_A^!B$ and $q_B^!A$, subject to the conditions of §4. It does not seem evident that such a pair of structures is equivalent to a pair of representations satisfying the twisted derivation conditions (together with a third which is vacuous when M is a point; see 6.1), and thus in turn equivalent to a single Lie algebroid structure on the Whitney sum $A \oplus B$ as in 6.2. We prove this in 6.7.

It then follows in Theorem 6.9 that a matched pair structure on Lie algebroids A, B with given representations ρ, σ is equivalent to a Lie bialgebroid structure on $(A \ltimes B^*, A^* \ltimes B)$, where the semi-direct products are formed with respect to the contragredient representations. Thus the single Lie bialgebroid equation, applied to these semi-direct products, also specializes to the system of matched pair equations.

One immediately striking feature of the notion of Lie bialgebra is the number of different perspectives from which it can be understood — as a pair of compatible actions, as a Manin triple, as the infinitesimal structure of a Poisson group. We show in this paper that these different approaches, extended to general Lie algebroids and encompassing a very much broader range of phenomena, can be unified into the single concept of double Lie algebroid. Because the bialgebroid equation is the key compatibility condition for a double Lie algebroid, and may be formulated diagrammatically, all of the notions subsumed under that of double Lie algebroid should be capable of formulation in fairly general categories, including those arising in modern forms of quantization.

Our definition of double Lie algebroid depends crucially on the underlying duality for finite rank vector bundles. It would be very interesting to be able to define a comparable concept of double for Lie algebroid-like structures on general modules (variously known as Lie pseudoalgebras, Gerstenhaber algebras, Lie–Rinehart algebras, and a number of other terminologies).

A subsequent article [25] extends to actions of Poisson groupoids Lu’s construction [16] of a matched pair of Lie algebroids for a Poisson group action. In the general case of Poisson groupoid actions, this produces a double Lie algebroid which does not correspond to either a matched pair or a Lie bialgebroid.

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2 DOUBLE VECTOR BUNDLES AND THEIR DUALITY

We begin by recalling the duality of double vector bundles from [22] or [11]. We then show that the existence of the duality between the duals of a double vector bundle reflects the existence of triple structures associated with its tangent and cotangent. The use of these triple structures shortens many arguments throughout the paper.

The definition of *double vector bundle* given in the Introduction (see Figure 1(a)) is complete but needs elaboration. The commutativity conditions stated there are precisely what is needed to ensure that when four elements ξ_1, \dots, ξ_4 of \mathcal{A} are such that the LHS of

$$(\xi_1 \overset{+}{_H} \xi_2) \overset{+}{_V} (\xi_3 \overset{+}{_H} \xi_4) = (\xi_1 \overset{+}{_V} \xi_3) \overset{+}{_H} (\xi_2 \overset{+}{_H} \xi_4)$$

is defined, then the RHS is also, and they are equal. Here $\overset{+}{_H}$ denotes the addition in $\mathcal{A} \longrightarrow A^V$ and $\overset{+}{_V}$ the addition in $\mathcal{A} \longrightarrow A^H$. See [30] or [20, §1].

We denote the projections in Figure 1(a) by $\tilde{q}_H: \mathcal{A} \longrightarrow A^V$ and $\tilde{q}_V: \mathcal{A} \longrightarrow A^H$; these are vector bundle morphisms over the projections $q_H: A^H \longrightarrow M$ and $q_V: A^V \longrightarrow M$ respectively. The intersection of the kernels of the two projections defined on \mathcal{A} is called the *core* of the double vector bundle \mathcal{A} and denoted K . The vector bundle structures on \mathcal{A} induce a common vector bundle structure on K with base M . The kernel of $\tilde{q}_H: \mathcal{A} \longrightarrow A^V$ is now the pullback $q_H^! K$ of K across q_H , and the kernel of $\tilde{q}_V: \mathcal{A} \longrightarrow A^H$ is $q_V^! K$. In practice cores are often identified with structures which are not strictly subsets of \mathcal{A} and in such cases we write \bar{k} for the element of \mathcal{A} corresponding to a $k \in K$.

Given $X \in A^H$ we denote the zero of $\mathcal{A} \longrightarrow A^H$ above X by $\tilde{0}_X^V$. Similarly the zero of $\mathcal{A} \longrightarrow A^V$ above $x \in A^V$ is denoted $\tilde{0}_x^H$.

Consider the dual \mathcal{A}^{*V} of the vertical bundle structure $\mathcal{A} \longrightarrow A^H$. In addition to its standard structure on base A^H , this has a vector bundle structure on base K^* , the projection of which is defined by

$$\langle \tilde{q}_V^{(*)}(\Phi), k \rangle = \langle \Phi, \tilde{0}_X^V \mathbin{\mathrel{+}_H} \bar{k} \rangle \quad (1)$$

where $\Phi: \tilde{q}_V^{-1}(X) \longrightarrow \mathbb{R}$, $X \in A_m^H$, and $k \in K_m$. The addition in $\mathcal{A}^{*V} \longrightarrow K^*$, which we also denote by $\mathbin{\mathrel{+}_H}$, is defined by

$$\langle \Phi \mathbin{\mathrel{+}_H} \Phi', \xi \mathbin{\mathrel{+}_H} \xi' \rangle = \langle \Phi, \xi \rangle + \langle \Phi', \xi' \rangle. \quad (2)$$

The zero of $\mathcal{A}^{*V} \longrightarrow K^*$ above $\kappa \in K_m^*$ is denoted $\tilde{0}_\kappa^{(*V)}$ and defined by

$$\langle \tilde{0}_\kappa^{(*V)}, \tilde{0}_x^H \mathbin{\mathrel{+}_V} \bar{k} \rangle = \langle \kappa, k \rangle$$

where $x \in A_m^V$, $k \in K_m$. The scalar multiplication is defined in a similar way. These two structures make \mathcal{A}^{*V} into a double vector bundle as in Figure 2(a), the *vertical dual* of \mathcal{A} .

$$\begin{array}{ccc} \mathcal{A}^{*V} & \xrightarrow{\tilde{q}_V^{(*)}} & K^* \\ \tilde{q}_{*V} \downarrow & & \downarrow q_{K^*} \\ A^H & \xrightarrow{q_H} & M \end{array} \quad (a)$$

$$\begin{array}{ccc} \mathcal{A}^{*H} & \xrightarrow{\tilde{q}_{*H}} & A^V \\ \tilde{q}_H^{(*)} \downarrow & & \downarrow q_V \\ K^* & \xrightarrow{q_{K^*}} & M \end{array} \quad (b)$$

Figure 2:

The core of \mathcal{A}^{*V} identifies with $(A^V)^*$, with the core element $\bar{\psi}$ corresponding to $\psi \in (A_m^V)^*$ given by

$$\langle \bar{\psi}, \tilde{0}_x^H \mathbin{\mathrel{+}_V} \bar{k} \rangle = \langle \psi, x \rangle.$$

See [31] or [22, §3].

There is also a *horizontal dual* \mathcal{A}^{*H} with sides A^V and K^* and core $(A^H)^*$, as in Figure 2(b). There is now the following somewhat surprising result.

Theorem 2.1 ([22, 3.1], [11, Thm. 16]) *There is a natural (up to sign) duality between $\mathcal{A}^{*V} \longrightarrow K^*$ and $\mathcal{A}^{*H} \longrightarrow K^*$ given by*

$$\langle \Phi, \Psi \rangle = \langle \Psi, \xi \rangle - \langle \Phi, \xi \rangle \quad (3)$$

where $\Phi \in \mathcal{A}^{*V}$, $\Psi \in \mathcal{A}^{*H}$ have $\tilde{q}_V^{(*)}(\Phi) = \tilde{q}_H^{(*)}(\Psi)$ and ξ is any element of \mathcal{A} with $\tilde{q}_V(\xi) = \tilde{q}_V^*(\Phi)$ and $\tilde{q}_H(\xi) = \tilde{q}_H^*(\Psi)$.

The pairing on the LHS of (3) is over K^* , whereas the pairings on the RHS are over A^V and A^H respectively.

Example 2.2 Given an ordinary vector bundle (A, q, M) there is the tangent double vector bundle of Figure 1(b); for a detailed account of this see [1] or [26, §5]. The core of TA consists of the vertical vectors along the zero section and identifies canonically with A . Dualizing the standard structure $TA \longrightarrow A$ gives the cotangent double vector bundle of Figure 1(c) with core T^*M . Dualizing the structure $TA \longrightarrow TM$ gives a double vector bundle which we denote $(T^\bullet A; A^*, TM; M)$; this is canonically isomorphic to $(T(A^*); A^*, TM; M)$ under an isomorphism $I: T(A^*) \longrightarrow T^\bullet A$ given by

$$\langle I(\mathcal{X}), \xi \rangle = \langle \mathcal{X}, \xi \rangle$$

where $\mathcal{X} \in T(A^*)$ and $\xi \in TA$ have $T(q_*)(\mathcal{X}) = T(q)(\xi)$, and $\langle \langle \cdot, \cdot \rangle \rangle$ is the tangent pairing of $T(A^*)$ and TA over TM . See [26, §5].

The structure of $\mathcal{A} = TA$ thus induces a pairing of T^*A and $T(A^*)$ over A^* given by

$$\langle \Phi, \mathcal{X} \rangle = \langle \mathcal{X}, \xi \rangle - \langle \Phi, \xi \rangle$$

where $\Phi \in T^*A$ and $\mathcal{X} \in T(A^*)$ have $r(\Phi) = p_{A^*}(\mathcal{X})$, and $\xi \in TA$ is chosen so that $T(q)(\xi) = T(q_*)(\mathcal{X})$ and $p_A(\xi) = c_A(\Phi)$. (Here c_A is the projection of the standard cotangent bundle and $r: T^*A \longrightarrow A^*$ is a particular case of (1).) This pairing is nondegenerate by a general result [22, 3.1] so it defines an isomorphism of double vector bundles $R: T^*A^* \longrightarrow T^*A$ by the condition

$$\langle R(\mathcal{F}), \mathcal{X} \rangle = \langle \mathcal{F}, \mathcal{X} \rangle$$

where the pairing on the RHS is the standard one of $T^*(A^*)$ and $T(A^*)$ over A^* . This R preserves the side bundles A and A^* but induces $-\text{id}: T^*M \longrightarrow T^*M$ as the map of cores. It is an antisymplectomorphism with respect to the exact symplectic structures. In summary we now have the very useful equation

$$\langle \mathcal{F}, \mathcal{X} \rangle + \langle R(\mathcal{F}), \xi \rangle = \langle \mathcal{X}, \xi \rangle, \quad (4)$$

for $\mathcal{F} \in T^*A^*$, $\mathcal{X} \in T(A^*)$, $\xi \in TA$, where the pairings are over A^* , A and TM respectively. For all of this see [26, §5].

Now return to the general double vector bundle in Figure 1(a), denoting the core by K . Each vector bundle structure on \mathcal{A} gives rise to a double cotangent bundle. These two double cotangents fit together into a triple structure as in Figure 3(a). This structure is a triple vector bundle in an obvious sense: each face is a double vector bundle and for each vector bundle structure on $T^*\mathcal{A}$, the maps defining the structure are morphisms of double vector

bundles. The left and rear faces of Figure 3(a) are the two cotangent doubles of the two structures on \mathcal{A} and the top face may be regarded as the cotangent double of either of the two duals of \mathcal{A} . (In all diagrams of this type, we take the oblique arrows to be coming out of the page.) The fact that the cube in Figure 3(a) may be rotated about the diagonal from $T^*\mathcal{A}$ to M , preserving its type, is essentially a consequence of Theorem 2.1.

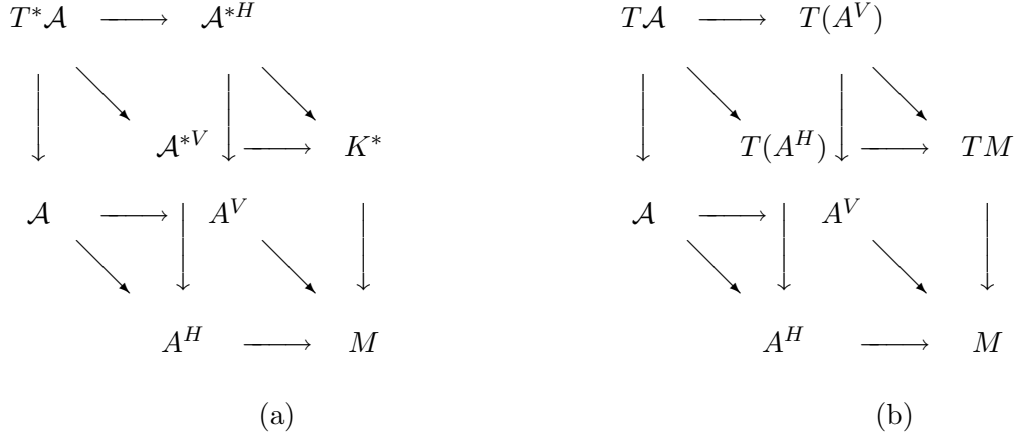


Figure 3:

Figure 3(a) is, in a sense we will make precise elsewhere, the vertical dual of the tangent prolongation of \mathcal{A} as given in Figure 3(b). Five of the six faces in 3(a) are double vector bundles of types considered already; it is only necessary to verify that the top face is a double vector bundle. The cores of the same five faces are known, and we take the core of the top face to be T^*K . (This is a special case of [22, 1.5].) Taking these cores in pairs, together with a parallel edge, then gives three double vector bundles: the left–right cores form $(T^*(A^H); A^H, (A^H)^*; M)$, the back–front cores form $(T^*(A^V); A^V, (A^V)^*; M)$ and the up–down cores form $(T^*K; K, K^*; M)$. Each of these *core double vector bundles* has core T^*M .

We will also need to consider the cotangent triples of the two duals of \mathcal{A} . Figure 4(a) is the cotangent triple of the double vector bundle $(\mathcal{A}^{*V}; A^H, K^*; M)$; the † denotes the dual over K^* . We use the isomorphisms of double vector bundles

$$Z_V: (\mathcal{A}^{*H})^\dagger \longrightarrow \mathcal{A}^{*V}, \quad Z_H: (\mathcal{A}^{*V})^\dagger \longrightarrow \mathcal{A}^{*H}$$

induced by the pairing (3). Note that Z_V preserves both sides, A^H and K^* , but induces $-\text{id}: (A^V)^* \longrightarrow (A^V)^*$ on the cores, while Z_H preserves K^* and the core $(A^H)^*$, but induces $-\text{id}$ on the sides A^V ; this reflects the fact that $Z_V = Z_H^\dagger$, the dual over K^* ; see [22, 3.6].

When $\mathcal{A} = T^*A$ as in Figure 1(b), we have $Z_V = R \circ I^\dagger$, where the † dual here is over A^* .

3 PRELIMINARY CASE

Before addressing the general concept of double Lie algebroid, it will be useful to deal with a very special case.

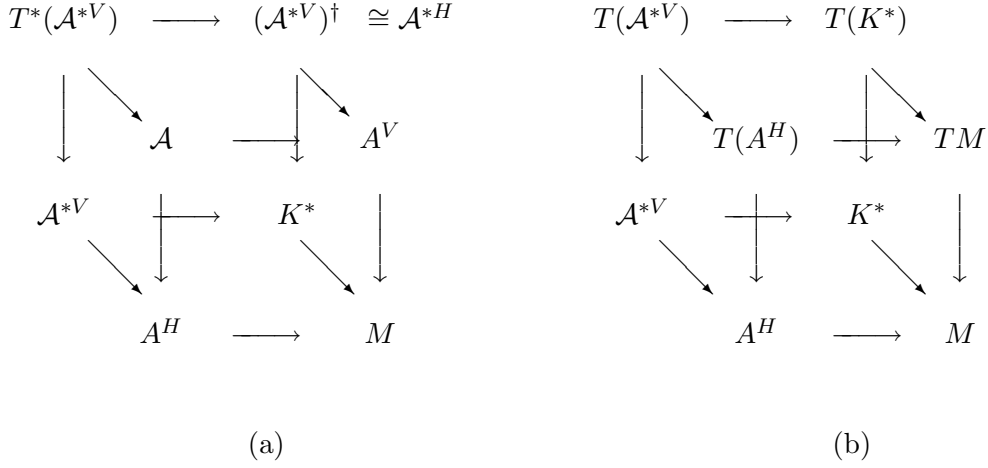


Figure 4:

Definition 3.1 An \mathcal{LA} -vector bundle is a double vector bundle as in Figure 1(a) together with Lie algebroid structures on a pair of parallel sides, such that the structure maps of the other pair of vector bundle structures are Lie algebroid morphisms.

For definiteness, take the Lie algebroid structures to be on $\mathcal{A} \rightarrow A^H$ and $A^V \rightarrow M$.

In the terminology of [20, §4], an \mathcal{LA} -vector bundle is an \mathcal{LA} -groupoid in which the groupoid structures are vector bundles (and in which the scalar multiplication also preserves the Lie algebroid structures). The core of a general \mathcal{LA} -groupoid has a Lie algebroid structure induced from the Lie algebroid structure on \mathcal{A} [20, §5]. Namely, each $k \in \Gamma K$ induces $\bar{k} \in \Gamma_{A^H} \mathcal{A}$ defined by $\bar{k}(X) = k(m) +_H \tilde{0}_X^V$ for $X \in A_m^H$ and the bracket on ΓK is obtained by $[\bar{k}, \bar{\ell}] = [\bar{k}, \bar{\ell}]$.

Lemma 3.2 The anchor and the bracket on the core of an \mathcal{LA} -vector bundle are zero.

PROOF. The anchor $\tilde{a}_V : \mathcal{A} \rightarrow T(A^H)$ is a morphism of double vector bundles and therefore induces a core map $\partial_H : K \rightarrow A^H$ which, by [20, §5], is a Lie algebroid morphism. Since A^H is abelian, we have $a_K = a_H \circ \partial_H = 0$.

Horizontal scalar multiplication by $t \neq 0$ defines a morphism $\mathcal{A} \rightarrow \mathcal{A}$ over $A^H \rightarrow A^H$ and therefore induces a map of sections $t_H : \Gamma_{A^H} \mathcal{A} \rightarrow \Gamma_{A^H} \mathcal{A}$; the Lie algebroid condition then ensures that $[t_H(\xi), t_H(\eta)] = t_H([\xi, \eta])$ for all $\xi, \eta \in \Gamma_{A^H} \mathcal{A}$. Now for $k \in \Gamma K$, $\bar{t}k = t_H(\bar{k})$ and so $t[k, \bar{\ell}] = [t_H(\bar{k}), t_H(\bar{\ell})] = [\bar{t}k, \bar{t}\bar{\ell}] = t^2[k, \bar{\ell}]$. Therefore the bracket must be zero. ■

Since $\mathcal{A} \rightarrow A^H$ is a Lie algebroid, the dual \mathcal{A}^{*V} has its dual Poisson structure, which is linear with respect to the bundle structure over A^H . The remainder of this section gives the proof of the following result.

Theorem 3.3 The Poisson structure on \mathcal{A}^{*V} is also linear with respect to the bundle structure over K^* .

This is actually a special case of [22, 3.14] (providing attention is paid to the scalar multiplication). We give a direct proof however, since there are special features which are needed later.

The functions $\mathcal{A}^{*V} \rightarrow \mathbb{R}$ which are linear with respect to K^* are determined by sections of $\mathcal{A}^{*H} \rightarrow K^*$ via the duality (3) which we here denote by $\langle \cdot, \cdot \rangle_{K^*}$ for clarity. Given $\Psi \in \Gamma_{K^*} \mathcal{A}^{*H}$ define

$$\ell_\Psi^\dagger: \mathcal{A}^{*V} \rightarrow \mathbb{R}, \quad \Phi \mapsto \langle \Phi, \Psi(\tilde{q}_V^{(*)}(\Phi)) \rangle_{K^*}.$$

There are two principal cases. Firstly, consider sections $\xi \in \Gamma_{A^H} \mathcal{A}$, $x \in \Gamma A^V$. The pair (ξ, x) is a (vertical) linear section if (ξ, x) is a vector bundle morphism from (A^H, q_H, M) to $(\mathcal{A}, \tilde{q}_H, A^V)$.

Proposition 3.4 (i) If (ξ, x) is a linear section, then $\ell_\xi: \mathcal{A}^{*V} \rightarrow \mathbb{R}$ is linear with respect to K^* as well as A^H , and the restriction of ℓ_ξ to the core of \mathcal{A}^{*V} is $\ell_x: (A^V)^* \rightarrow \mathbb{R}$.

(ii) If the function ℓ_ξ defined by some $\xi \in \Gamma_{A^H} \mathcal{A}$ is linear with respect to K^* as well as A^H , then the restriction of ℓ_ξ to the core defines a section $x \in \Gamma A^V$, and (ξ, x) is a linear section.

PROOF. (i) Take elements $(\Phi_1; X_1, \kappa; m)$ and $(\Phi_2; X_2, \kappa; m)$ of \mathcal{A}^{*V} . Their horizontal sum is of the form $(\Phi_1 \overset{+}{_H} \Phi_2; X_1 + X_2, \kappa; m)$ and so, using first the linearity of ξ and then the definition of $\overset{+}{_H}$,

$$\begin{aligned} \ell_\xi(\Phi_1 \overset{+}{_H} \Phi_2) &= \langle \Phi_1 \overset{+}{_H} \Phi_2, \xi(X_1 + X_2) \rangle = \langle \Phi_1 \overset{+}{_H} \Phi_2, \xi(X_1) \overset{+}{_H} \xi(X_2) \rangle \\ &= \langle \Phi_1, \xi(X_1) \rangle + \langle \Phi_2, \xi(X_2) \rangle = \ell_\xi(\Phi_1) + \ell_\xi(\Phi_2). \end{aligned}$$

The remainder of (i) is similar.

(ii) Since the core embedding $(A^V)^* \rightarrow \mathcal{A}^{*V}$ is linear with respect to either structure on \mathcal{A}^{*V} , the first statement is immediate. The relationship between ξ and x is $\langle \xi(0_m^H), \bar{\psi} \rangle = \langle x(m), \psi \rangle$ for all $\psi \in (A_m^V)^*$. Writing $y = \tilde{q}_H(\xi(0_m^H))$ we therefore have $\xi(0_m^H) = \tilde{0}_y^H \overset{+}{_V} \bar{k}$ for some $k \in K_m$. Pairing with any $\bar{\psi}$ we find $y = x(m)$.

Linearity over K^* means that, for any $(\Phi_1; X_1, \kappa; m), (\Phi_2; X_2, \kappa; m)$ in \mathcal{A}^{*V} ,

$$\langle \xi(X_1 + X_2), \Phi_1 \overset{+}{_H} \Phi_2 \rangle = \langle \xi(X_1), \Phi_1 \rangle + \langle \xi(X_2), \Phi_2 \rangle. \quad (5)$$

Putting $X_1 = X_2 = 0_m^H$ and $\Phi_1 = \Phi_2 = \tilde{0}_\kappa^{(*V)}$ for any $\kappa \in K_m^*$, we have $\langle \xi(0_m^H), \tilde{0}_\kappa^{(*V)} \rangle = 0$. Using $\xi(0_m^H) = \tilde{0}_{x(m)}^H \overset{+}{_V} \bar{k}$, this gives $\langle \kappa, k \rangle = 0$ for all κ , so $k = 0$.

Next in (5) put $\Phi_1 = \bar{\psi}$, $\Phi_2 = \tilde{0}_X$ and $X_1 = 0_m^H$, $X_2 = X$. Then we have

$$\begin{aligned} \langle \psi, x(m) \rangle &= \langle \xi(X), \bar{\psi} \overset{+}{_H} \tilde{0}_X \rangle = \langle \tilde{0}_y^H \overset{+}{_H} \xi(X), \bar{\psi} \overset{+}{_H} \tilde{0}_X \rangle \\ &= \langle \tilde{0}_y^H, \bar{\psi} \rangle + \langle \xi(X), \tilde{0}_X \rangle = \langle \psi, y \rangle + 0 \end{aligned}$$

where $y = \tilde{q}_H(\xi(X))$. So $x(q_H(X)) = x(m) = y = \tilde{q}_H(\xi(X))$. The proof that (ξ, x) is linear is now immediate. ■

Given a linear section (ξ, x) , denote by ξ^\square the section of $\mathcal{A}^{*H} \rightarrow K^*$ which ℓ_ξ defines. Thus

$$\langle \Phi, \xi^\square(\kappa) \rangle_{K^*} = \ell_\xi(\Phi) = \langle \Phi, \xi(X) \rangle_{A^H}$$

for $(\Phi; X, \kappa; m)$ in \mathcal{A}^{*V} . The proof of the result which follows is not difficult.

Proposition 3.5 (ξ^\square, x) is a (vertical) linear section of \mathcal{A}^{*H} .

There is thus a bijective correspondence between vertical linear sections of \mathcal{A} and vertical linear sections of \mathcal{A}^{*H} . This of course applies to any double vector bundle: in the case where \mathcal{A} is the tangent of an ordinary vector bundle A , as in Figure 1(b), this is the correspondence between linear vector fields on A and linear vector fields on A^* (see [27, §2]).

Secondly, any $\varphi \in \Gamma(A^H)^*$ induces a core section $\overline{\varphi}$ of $\mathcal{A}^{*H} \rightarrow K^*$ by

$$\overline{\varphi}(\kappa) = \widetilde{0}_\kappa^{(*H)} \mathrel{+}_H \overline{\varphi(m)}, \quad \kappa \in K_m^*,$$

which induces $\ell_\varphi^\dagger: \mathcal{A}^{*V} \rightarrow \mathbb{R}$. A simple calculation shows that

$$\ell_\varphi^\dagger = \ell_\varphi \circ \widetilde{q}_{*V}.$$

We also need the pullbacks across $\mathcal{A}^{*V} \rightarrow K^*$ of functions on K^* . In particular, for $k \in \Gamma K$ and $f \in C(M)$, we have

$$\ell_k \circ \widetilde{q}_V^{(*)} = \ell_{\overline{k}}, \quad f \circ q_{K^*} \circ \widetilde{q}_V^{(*)} = f \circ q_H \circ \widetilde{q}_{*V}$$

where $\overline{k} \in \Gamma_{AH} \mathcal{A}$ is the core section for k .

We now have four classes of functions $\mathcal{A}^{*V} \rightarrow \mathbb{R}$. The functions linear over K^* are generated by the ℓ_ξ for (ξ, x) linear, together with the $\ell_\varphi \circ \widetilde{q}_{*V}$ for $\varphi \in \Gamma(A^H)^*$. The pullbacks from K^* are generated by the $\ell_{\overline{k}}$, $k \in \Gamma K$, together with the $f \circ q_H \circ \widetilde{q}_{*V}$ for $f \in C(M)$. In the proof of 3.3 we therefore need only consider functions of these types; we refer to them as types L_1, L_2, P_1, P_2 respectively.

Lemma 3.6 If (ξ, x) and (η, y) are linear sections, then $([\xi, \eta], [x, y])$ is also.

PROOF. For a section ξ which projects under \widetilde{q}_H to a section x , define a section $\xi \oplus \xi$ of $\mathcal{A} \oplus_{AV} \mathcal{A} \rightarrow A^H \oplus_M A^H$ by $(\xi \oplus \xi)(X \oplus Y) = \xi(X) \oplus \xi(Y)$. Then (ξ, x) is linear if and only if $\xi \oplus \xi$ projects to ξ under the horizontal addition $\mathrel{+}_H: \mathcal{A} \oplus_{AV} \mathcal{A} \rightarrow \mathcal{A}$, which is a Lie algebroid morphism over $+: A^H \oplus_M A^H \rightarrow A^H$ by hypothesis. Since the two components of a $\xi \oplus \xi$ depend on each variable separately, we have $[\xi \oplus \xi, \eta \oplus \eta] = [\xi, \eta] \oplus [\xi, \eta]$ and the result follows. ■

From $\{\ell_\xi, \ell_\eta\} = \ell_{[\xi, \eta]}$ it follows that the bracket of two type L_1 functions is L_1 . The bracket of two L_2 functions is zero, since they are pullbacks from A^H . For the mixed case,

$$\{\ell_\xi, \ell_\varphi \circ \widetilde{q}_{*V}\} = \widetilde{a}_V(\xi)(\ell_\varphi) \circ \widetilde{q}_{*V}.$$

Now $\widetilde{a}_V(\xi)$ is a linear vector field on A^H so, by [27, (5)], we can define

$$\widetilde{a}_V(\xi)(\ell_\varphi) = \ell_{D_\xi^{(*)}(\varphi)}$$

where $D_\xi^{(*)}: \Gamma(A^H)^* \rightarrow \Gamma(A^H)^*$. In particular the bracket of an L_1 and an L_2 is an L_2 .

Here $D^{(*)}$ is a covariant differential operator in the sense of [19, III§2]; that is, $D = D^{(*)}$ is a linear differential operator of order ≤ 1 and there is a vector field $X = a_V(\xi)$ such that $D(f\varphi) = fD(\varphi) + X(f)\varphi$ for all $f \in C(M)$ and section φ . For any vector bundle E there is

a vector bundle $\text{CDO}(E)$ the sections of which are the covariant differential operators, and with anchor $D \mapsto X$ and the usual bracket, $\text{CDO}(E)$ is a Lie algebroid.

For an L_1 and a P_1 we have $\{\ell_\xi, \ell_{\bar{k}}\} = \ell_{[\xi, \bar{k}]}$. Now $\xi \sim x$ and $\bar{k} \sim 0$ under \tilde{q}_H , so $[\xi, \bar{k}] \sim 0$. It is therefore \bar{k}' for some $k' \in \Gamma K$ which we denote $C_\xi(k)$; it is easily checked that $C_\xi: \Gamma K \rightarrow \Gamma K$ is again a covariant differential operator. Thus we have another P_1 .

For an L_1 and a P_2 we have

$$\{\ell_\xi, f \circ q_H \circ \tilde{q}_{*V}\} = \tilde{a}_V(\xi)(f \circ q_H) \circ \tilde{q}_{*V} = x(f) \circ q_H \circ \tilde{q}_{*V},$$

again a P_2 , and for an L_2 and a P_1 ,

$$\{\ell_\varphi \circ \tilde{q}_{*V}, \ell_{\bar{k}}\} = -\tilde{a}_V(\bar{k})(\ell_\varphi) \circ \tilde{q}_{*V} = -\partial_H(k)^\dagger(\ell_\varphi) \circ \tilde{q}_{*V} = -\langle \partial_H(k), \varphi \rangle \circ q_H \circ \tilde{q}_{*V},$$

a P_2 . Here z^\dagger , for a vector field z , is the vertical lift of z as in [27, §2]. The remaining four cases give zero, using 3.2.

This completes the proof of Theorem 3.3. \blacksquare

Since the Poisson structure is linear over A^H , $\pi^{\#V}: T^*(\mathcal{A}^{*V}) \rightarrow T(\mathcal{A}^{*V})$ is a morphism of double vector bundles for the left faces of Figure 4, with the corner map $\mathcal{A} \rightarrow T(A^H)$ being \tilde{a}_V and core map $-\tilde{a}_V^*: T^*(A^H) \rightarrow \mathcal{A}^{*V}$. From 3.3 it now follows that $\pi^{\#V}$ is also a morphism of double vector bundles for the rear faces of Figure 4. Denote the corner map $(\mathcal{A}^{*V})^\dagger \rightarrow T(K^*)$ by χ_V ; since $\pi^{\#V}$ is skew-symmetric, the core map of the rear faces is $-\chi_V^*: T^*K^* \rightarrow \mathcal{A}^{*V}$. It then follows by a simple argument (as in [22, 2.3]) that $\pi^{\#V}$ is a morphism of triple vector bundles.

4 ABSTRACT DOUBLE LIE ALGEBROIDS

We now turn to the general notion of double Lie algebroid. Again consider a double vector bundle as in Figure 1(a). We now assume that there are Lie algebroid structures on all four sides. The definition comprises three conditions.

Condition I

With respect to the two vertical Lie algebroids, $\mathcal{A} \rightarrow A^H$ and $A^V \rightarrow M$, the double vector bundle \mathcal{A} is an \mathcal{LA} -vector bundle. Likewise, with respect to the two horizontal Lie algebroids, $\mathcal{A} \rightarrow A^V$ and $A^H \rightarrow M$, the double vector bundle \mathcal{A} is an \mathcal{LA} -vector bundle.

Denote the four anchors by $\tilde{a}_V: \mathcal{A} \rightarrow T(A^H)$, $\tilde{a}_H: \mathcal{A} \rightarrow T(A^V)$, and $a_V: A^V \rightarrow TM$, $a_H: A^H \rightarrow TM$. As usual we denote all four brackets by $[\cdot, \cdot]$; the notation for elements will make clear which structure we are using.

The anchors thus give morphisms of double vector bundles

$$(\tilde{a}_V; \text{id}, a_V; \text{id}): (\mathcal{A}; A^H, A^V; M) \rightarrow (T(A^H); A^H, TM; M),$$

$$(\tilde{a}_H; a_H, \text{id}; \text{id}): (\mathcal{A}; A^H, A^V; M) \rightarrow (T(A^V); TM, A^V; M)$$

and so define morphisms of their cores; denote these by $\partial_H: K \rightarrow A^H$ and $\partial_V: K \rightarrow A^V$.

Now return to $\pi^{\#V}$ and Figure 4. Since the corner map $\mathcal{A} \rightarrow T(A^H)$ is \tilde{a}_V , the corner map $A^V \rightarrow TM$ is a_V (or $-a_V$ if Z_H is incorporated) and the core map for the front faces

is ∂_H . Likewise, since the core map of the left faces is $-\tilde{a}_V^*$, the core map of the right faces must be $-\partial_H^*: (A^H)^* \rightarrow K^*$ (whether or not Z_H is incorporated). Lastly, the core map of the top faces is $\pi_V^\#: T^*((A^V)^*) \rightarrow T((A^V)^*)$, the anchor for the Poisson structure on $(A^V)^*$ dual to the given Lie algebroid structure on A^V . (These observations are all special cases of [22, §3].)

Each of the maps of the core double vector bundles induces on $T^*M \rightarrow (A^V)^*$ the map $-a_V^*$.

Similarly we can analyze $\pi^{\#H}: T^*(\mathcal{A}^{*H}) \rightarrow T(\mathcal{A}^{*H})$ as a morphism of triple vector bundles, and obtain $\chi_H: (\mathcal{A}^{*H})^\dagger \rightarrow T(K^*)$.

For Condition II, note first that it is automatic that \tilde{a}_V is a morphism of Lie algebroids over A^H and that a_V is a morphism of Lie algebroids over M .

Condition II

The anchors \tilde{a}_V and a_V form a morphism of Lie algebroids (\tilde{a}_V, a_V) with respect to the horizontal structure on \mathcal{A} and the prolongation to $TA^H \rightarrow TM$ of the structure on $A^H \rightarrow M$. Likewise, the anchors \tilde{a}_H and a_H form a morphism of Lie algebroids with respect to the vertical structure on \mathcal{A} and the prolongation to $TA^V \rightarrow TM$ of the structure on $A^V \rightarrow M$.

By Condition I and §3, the Poisson structure on $\mathcal{A}^{*H} \rightarrow K^*$ is linear, and therefore induces a Lie algebroid structure on its dual $(\mathcal{A}^{*H})^\dagger$. We use Z_V to transfer this to $\mathcal{A}^{*V} \rightarrow K^*$. Similarly the linear Poisson structure on $\mathcal{A}^{*V} \rightarrow K^*$ induces a Lie algebroid structure on $(\mathcal{A}^{*V})^\dagger \rightarrow K^*$.

Condition III

*With respect to these structures, $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is a Lie bialgebroid. Further, $(\mathcal{A}^{*V}; A^H, K^*; M)$ is an \mathcal{LA} -vector bundle with respect to the horizontal Lie algebroid structures and likewise $(\mathcal{A}^{*H}; K^*, A^V; M)$ is an \mathcal{LA} -vector bundle with respect to the vertical structures.*

Definition 4.1 *A double Lie algebroid is a double vector bundle as in Figure 1(a) equipped with Lie algebroid structures on all four sides such that the above conditions I, II, III are satisfied.*

For $(\mathcal{A}; A^H, A^V; M)$ a double Lie algebroid, we call $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ the *associated Lie bialgebroid*.

The notion of Lie bialgebroid was defined in [26] in terms of the coboundary operators associated to A and to A^* ; a more efficient and elegant reformulation was then given in [12]. The definition most useful to us here is quoted below in 5.1. For the moment we only need the following.

Suppose that (E, E^*) is a Lie bialgebroid on base P and denote the anchors by e and e_* . Then we take the Poisson structure on P to be $\pi_P^\# = e_* \circ e^*$; this is the opposite to [26], but the same as [12]. It follows that e is a Poisson map (to the tangent lift structure on TP) and e_* is anti-Poisson.

One expects the core of a double Lie algebroid to have a Lie algebroid structure induced from those on \mathcal{A} . However, as 3.2 shows, the straightforward embedding of K in terms of core

sections yields only the zero structure (see also [24]). Here we obtain the correct structure in terms of its dual.

The anchor of $(\mathcal{A}^{*V})^\dagger$ is χ_V , the appropriate corner map of the Poisson anchor for \mathcal{A}^{*V} . On the other hand, the anchor for \mathcal{A}^{*V} itself is $\chi_H \circ Z_V^{-1}$. So the Poisson anchor for K^* is

$$\pi_{K^*}^\# = \chi_V \circ (Z_V^{-1})^\dagger \circ \chi_H^\dagger.$$

Now $Z_V^\dagger = Z_H$ has side map $-\text{id}: A^V \longrightarrow A^V$ and χ_V has side map a_V . The side map of χ_H^\dagger is the dual of the core map $-\partial_V^*$ of χ_H . Thus the side map of $\pi_{K^*}^\#$ is $a_V \circ \partial_V$. One likewise checks that the core map is $-\partial_H^* \circ a_H^*$. This proves the first half of the following result.

Proposition 4.2 *The anchor a_K for the Lie algebroid structure on K induced by the Poisson structure on K^* which arises from the Lie bialgebroid structure on $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is $a_V \circ \partial_V = a_H \circ \partial_H$. The maps $\partial_H: K \longrightarrow A^H$ and $\partial_V: K \longrightarrow A^V$ are Lie algebroid morphisms.*

PROOF. Since χ_V is the anchor for $(\mathcal{A}^{*V})^\dagger$, it is anti-Poisson into $T(K^*)$. Regarding χ_V as a morphism of the right faces in Figure 4, its core is $-\partial_H^*$, which is therefore anti-Poisson. So ∂^H is a morphism of Lie algebroids. ■

The most fundamental example motivating 4.1 is for us the double Lie algebroid of a double Lie groupoid, as constructed in [20], [24]. Some of what is required in order to verify that the double Lie algebroid of a double Lie groupoid does satisfy 4.1 has been given in [22], and we recall the details briefly.

In order to proceed, we need to describe the notion of double Lie groupoid in more detail (see [20] and references given there). A double Lie groupoid consists of a manifold S equipped with two Lie groupoid structures on bases H and V , each of which is a Lie groupoid over a common base M , such that the structure maps (source, target, multiplication, identity, inversion) of each groupoid structure on S are morphisms with respect to the other; see Figure 5(a). One should think of elements of S as squares, the horizontal edges of which

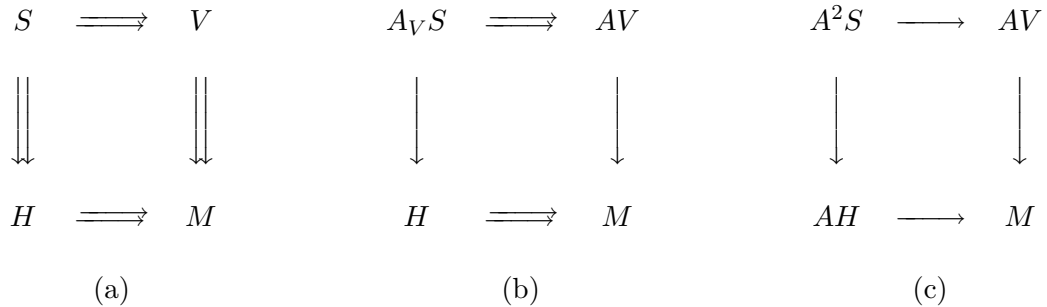


Figure 5:

come from H , the vertical edges from V , and the corner points from M .

Consider a double Lie groupoid $(S; H, V; M)$ as in Figure 5(a). Applying the Lie functor to the vertical structure $S \rightrightarrows H$ gives a Lie algebroid $A_V S \longrightarrow H$ which has also a groupoid

structure over AV obtained by applying the Lie functor to the structure maps of $S \rightrightarrows V$; this is the *vertical \mathcal{LA} -groupoid* of S [20, §4], as in Figure 5(b). The Lie algebroid of $A_V S \rightrightarrows AV$ is denoted $A^2 S$; there is a double vector bundle structure $(A^2 S; AH, AV; M)$ obtained by applying A to the vector bundle structure of $A_V S \rightarrow H$ [24]; see Figure 5(c). Reversing the order of these operations, one defines first the *horizontal \mathcal{LA} -groupoid* $(A_H S; AH, V; M)$ and then takes the Lie algebroid $A_2 S = A(A_H S)$. The canonical involution $J_S: T^2 S \rightarrow T^2 S$ then restricts to an isomorphism of double vector bundles $j_S: A^2 S \rightarrow A_2 S$ and allows the Lie algebroid structure on $A^2 S \rightarrow AV$ to be transported to $A_2 S \rightarrow AV$. Thus $A_2 S$ is a double vector bundle equipped with four Lie algebroid structures; in [24] we called this the *double Lie algebroid of S* . The core of both double vector bundles $A_2 S$ and $A^2 S$ is AC , the Lie algebroid of the core groupoid $C \rightrightarrows M$ of S [24, 1.6].

Consider $A_2 S$. The structure maps for the horizontal vector bundle $A_2 S \rightarrow AV$ are obtained by applying the Lie functor to the structure maps of $A_H S \rightarrow V$ and are therefore Lie algebroid morphisms with respect to the vertical Lie algebroid structures. The corresponding statement is true for the vertical vector bundle $A^2 S \rightarrow AH$ and this is transported by j_S to $A_2 S \rightarrow AH$. Thus Condition I holds.

Let $\overset{\vee}{a}_V: A_2 S \rightarrow TAH$ denote the anchor for the Lie algebroid of $A_H S \rightrightarrows AH$. Then, as with any Lie groupoid, $\overset{\vee}{a}_V = A(\overset{\vee}{\chi}_V)$ where $\overset{\vee}{\chi}_V: A_H S \rightarrow AH \times AH$ combines the target and source of $A_H S \rightrightarrows AH$. It is easily checked that $\overset{\vee}{\chi}_V$ is a morphism of \mathcal{LA} -groupoids over $\chi_V: V \rightarrow M \times M$ and $\text{id}: AH \rightarrow AH$, and so it follows, by using the methods of [24, §1], that $\overset{\vee}{a}_V$ is a morphism of Lie algebroids over a_V . Similarly one transports the result for the anchors $A^2 S \rightarrow TAV$ and $AH \rightarrow TM$. Thus Condition II is satisfied.

Now consider the bialgebroid condition. The vertical dual \mathcal{A}^{*V} is $A^*(A_H S)$ and in order to take the dual of this over A^*C we use the isomorphism $j'^V: A^*(A_H S) \rightarrow A(A_V^* S)$ of [22, (21)]. This induces $(\mathcal{A}^{*V})^\dagger \cong A^*(A_V^* S)$.

Now the structure on \mathcal{A}^{*V} itself comes from $(\mathcal{A}^{*H})^\dagger$. We have $\mathcal{A}^{*H} = A^\bullet(A_H S)$ and the isomorphism $I_H: A(A_H^* S) \rightarrow A^\bullet(A_H S)$ associated to the \mathcal{LA} -groupoid $A_H S$ in [22, §3] (see also (6) below) allows us to identify $(\mathcal{A}^{*H})^\dagger$ with $A^*(A_H^* S)$.

This proves that $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is isomorphic to $(A^*(A_H^* S), A^*(A_V^* S))$. Now the isomorphism $\mathcal{D}_H: A^*(A_H^* S) \rightarrow A(A_V^* S)$ of [22, 3.9] allows this to be written as $(A(A_V^* S), A^*(A_V^* S))$ and this is the Lie bialgebroid of $A_V^* S \rightrightarrows A^*C$, which was proved to be a Poisson groupoid in [22, 2.12]. Notice that we started with $A_2 S$, defined in terms of the horizontal \mathcal{LA} -groupoid, and ended with the Lie bialgebroid of the dual of the vertical \mathcal{LA} -groupoid.

To make this sketch into a proof, one must ensure that the various isomorphisms preserve the Poisson structures involved. Rather than do this, we prove a more general result.

Consider an \mathcal{LA} -groupoid as in Figure 6(a); that is, Ω is both a Lie algebroid over G and a Lie groupoid over A , and each of the groupoid structure maps is a Lie algebroid morphism; further, the map $\Omega \rightarrow A \times_M G$ defined by the source and the bundle projection, is a surjective submersion. Applying the Lie functor vertically gives a double vector bundle $\mathcal{A} = A\Omega$ as in Figure 6(b), with Lie algebroid structures on the vertical sides. It is shown in [24, §1] that the Lie algebroid structure of $\Omega \rightarrow G$ may be prolonged to $A\Omega \rightarrow AG$.

That the anchor $\overset{\vee}{a}: A\Omega \rightarrow TA$ for the Lie algebroid of $\Omega \rightrightarrows A$ is a morphism of Lie algebroids over $a_G: AG \rightarrow TM$ follows as in the case of $A_2 S$ above. The anchor $\mathbf{a}: A\Omega \rightarrow TAG$ for the prolongation structure is $j_G^{-1} \circ A(\tilde{a})$ where $j_G: TAG \rightarrow ATG$ is the canonical isomorphism of [26, 7.1]. Since $\tilde{a}: \Omega \rightarrow TG$ is a groupoid morphism over $a: A \rightarrow TM$,

$$\begin{array}{ccc}
\Omega & \xrightarrow{\tilde{q}} & G \\
\Downarrow & & \Downarrow \\
A & \xrightarrow{q_A} & M \\
& (a) &
\end{array}
\qquad
\begin{array}{ccc}
A\Omega & \xrightarrow{A(\tilde{q})} & AG \\
\downarrow \scriptstyle \vee q & & \downarrow \scriptstyle q_G \\
A & \xrightarrow{q_A} & M \\
& (b) &
\end{array}$$

Figure 6:

and j_G is an isomorphism of Lie algebroids over TM , it follows that Condition II is satisfied. Condition I is dealt with in the same way.

It was shown in [22, §3] that $\Omega^* \implies K^*$, the dual groupoid of Ω , together with the Poisson structures on Ω^* and K^* dual to the Lie algebroid structures on Ω and the core K , is a Poisson groupoid. Thus Condition III will follow from the next result.

Theorem 4.3 *The canonical isomorphism of double vector bundles $R = R^{gpd}$ from $A^*\Omega^*$ to $A^*\Omega = \mathcal{A}^{*V}$ is an isomorphism of Lie bialgebroids*

$$(A^*\Omega^*, \overline{A\Omega^*}) \longrightarrow (\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$$

where $(A\Omega^*, A^*\Omega^*)$ is the Lie bialgebroid of the Poisson groupoid Ω^* on K^* .

We first recall the map R from [22, 3.8]. Associated with $A\Omega$ oriented as in Figure 6(b) there is the pairing of the vertical and horizontal duals (3), which we write in mnemonic form:

$$\langle A^*\Omega, A^\bullet\Omega \rangle_{K^*} = \langle A^\bullet\Omega, A\Omega \rangle_{AG} - \langle A^*\Omega, A\Omega \rangle_A$$

with the subscripts indicating the bases of the pairings. As in [22, (19)], we use the canonical isomorphism $I: A\Omega^* \longrightarrow A^\bullet\Omega$ induced by the pairing $\langle\langle \cdot, \cdot \rangle\rangle: A\Omega^* \times_{AG} A\Omega \longrightarrow \mathbb{R}$ to transfer this to a pairing \dagger, \ddagger of $A^*\Omega$ and $A\Omega^*$ over K^* for which

$$\dagger A^*\Omega, A\Omega^* \ddagger = \langle\langle A\Omega^*, A\Omega \rangle\rangle - \langle A^*\Omega, A\Omega \rangle_A. \quad (6)$$

Now define R by $\dagger \mathcal{X}, R(\mathcal{F}) \ddagger = \langle \mathcal{X}, \mathcal{F} \rangle$, where $\mathcal{X} \in A\Omega^*, \mathcal{F} \in A^*\Omega^*$, and the pairing on the RHS is the standard one over K^* . We finally arrive at

$$\langle \mathcal{X}, \mathcal{F} \rangle_{K^*} + \langle R(\mathcal{F}), \Xi \rangle_A = \langle\langle \mathcal{X}, \Xi \rangle\rangle \quad (7)$$

for compatible $\Xi \in A\Omega$. Equivalently, $Z_V: (A^\bullet\Omega)^\dagger \longrightarrow A^*\Omega$ is given by

$$Z_V = R \circ I^\dagger. \quad (8)$$

The first part of the following result was stated without proof in [22, §3].

Proposition 4.4 (i) *The map $R: A^*\Omega^* \longrightarrow A^*\Omega$ is anti-Poisson from the Poisson structure dual to the Lie algebroid of $\Omega^* \rightrightarrows K^*$ to the Poisson structure dual to the Lie algebroid of $\Omega \rightrightarrows A$.*

(ii) *The map $I: A\Omega^* \longrightarrow A^\bullet\Omega$ is Poisson from the Poisson structure induced on $A\Omega^*$ [36] by the Poisson groupoid structure on $\Omega^* \rightrightarrows K^*$, to the Poisson structure dual to the prolonged Lie algebroid structure on $A\Omega \longrightarrow AG$.*

PROOF. It suffices [36] to prove that the graph of R is coisotropic in $A^*\Omega^* \times A^*\Omega$. Let $S = \Omega^* \times_G \Omega$ and write $F: S \longrightarrow \mathbb{R}$ for the pairing. Then F is a groupoid morphism, where S is the pullback groupoid over $K^* \times_M A$, and so, as in [22, 3.7], we can apply the Lie functor and get $\langle\langle \cdot, \cdot \rangle\rangle = A(F): AS \longrightarrow \mathbb{R}$. This is linear and so defines a section ν of the dual of AS , which is closed since $A(F)$ is a morphism. So by [27, 4.6], the image of ν is coisotropic.

It remains to show that the image of ν coincides with the graph of R . The image of ν consists of those $(\mathcal{F}, \mathcal{X}) \in A^*_\kappa\Omega \times A^*_Y\Omega$ such that

$$\langle(\mathcal{F}, \Phi), (\mathcal{X}, \Xi)\rangle = A(F)(\mathcal{X}, Xi)$$

for all $(\mathcal{X}, \Xi) \in AS$ compatible with (\mathcal{F}, Φ) . As in [26, 5.5], this equation expands to (7).

We leave the proof of (ii) to the reader. ■

PROOF OF THEOREM 4.3: We must first show that R is an isomorphism of Lie algebroids $A^*\Omega^* \longrightarrow \mathcal{A}^{*V}$. Now the Lie algebroid structure on \mathcal{A}^{*V} is induced from $(\mathcal{A}^{*H})^\dagger$ via Z_V . So what we have to show is that $Z_V^{-1} \circ R: A^*\Omega^* \longrightarrow (A^\bullet\Omega)^\dagger$ is an isomorphism of Lie algebroids, and this is equivalent to the dual over K^* being Poisson. This dual is, using (8), $I^{-1}: A^\bullet\Omega \longrightarrow A\Omega^*$, and so the result follows from 4.4(ii) above.

Secondly we must show that $R^\dagger: (\mathcal{A}^{*V})^\dagger \longrightarrow \overline{A\Omega^*}$ is an isomorphism of Lie algebroids over K^* . (Note that the minus sign is in the bundle over K^* .) This is equivalent to showing that the dual $R: \overline{A^*\Omega^*} \longrightarrow A^*\Omega$ is Poisson, and this is 4.4(i) above. ■

In summary, we have proved:

Theorem 4.5 *The double Lie algebroid $(A\Omega; A, AG; M)$ of an \mathcal{LA} -groupoid $(\Omega; A, G; M)$, as constructed in [24, §2], is a double Lie algebroid as defined in 4.1.*

In particular, the double Lie algebroids $(A_2S; AH, AV; M)$ and $(A^2S; AH, AV; M)$ of a double Lie groupoid $(S; H, V; M)$, as constructed in [24, §3], are double Lie algebroids as defined in 4.1.

Example 4.6 Let A be any Lie algebroid on M . Then $\Omega = A \times A$ has an \mathcal{LA} -groupoid structure over $M \times M$ and A , and the associated double Lie algebroid constructed in [24, §1] is $(TA; A, TM; M)$.

The associated duals are $\mathcal{A}^{*V} = T^*A$ and $\mathcal{A}^{*H} = T^\bullet A$. Using R and I as in [26], these can be identified with T^*A^* and $T(A^*)$, as bundles over A^* . The Lie algebroid structure on T^*A^* is the cotangent of the dual Poisson structure on A^* . The Lie algebroid structure on $T(A^*)$ is the standard tangent bundle structure. The Lie bialgebroid associated to TA is the standard Lie bialgebroid (T^*P, TP) for $P = A^*$.

Example 4.7 Taking $A = TM$ in the previous example, we see that T^2M is a double Lie algebroid, the associated bialgebroid of which is (T^*T^*M, TT^*M) . This is a Lie bialgebroid over T^*M , the induced Poisson structure being the standard symplectic structure.

The double Lie algebroids considered in the next two sections do not necessarily have an underlying \mathcal{LA} -groupoid.

5 THE DOUBLE LIE ALGEBROID OF A LIE BIALGEBROID

Here we use the following criterion for a Lie bialgebroid.

Theorem 5.1 [26, 6.2] *Let A be a Lie algebroid on M such that its dual vector bundle A^* also has a Lie algebroid structure. Denote their anchors by a, a_* . Then (A, A^*) is a Lie bialgebroid if and only if*

$$T^*(A^*) \xrightarrow{R} T^*(A) \xrightarrow{\pi_A^\#} TA \quad (9)$$

is a Lie algebroid morphism over a_ , where the domain $T^*(A^*) \rightarrow A^*$ is the cotangent Lie algebroid induced by the Poisson structure on A^* , and the target $TA \rightarrow TM$ is the tangent prolongation of A .*

Consider a Lie algebroid A on M together with a Lie algebroid structure on the dual, not *a priori* related to that on A . The structure on A^* induces a Poisson structure on A , and this gives rise to a cotangent Lie algebroid $T^*A \rightarrow A$. Equally, the Lie algebroid structure on A induces a Poisson structure on A^* and this gives rise to a cotangent Lie algebroid $T^*A^* \rightarrow A^*$. We transfer this latter structure to $T^*A \rightarrow A^*$ via R .

There are now four Lie algebroid structures on the four sides of $\mathcal{A} = T^*A$ as in Figure 1(c).

Theorem 5.2 *Let A be a Lie algebroid on M such that its dual vector bundle A^* also has a Lie algebroid structure. Then (A, A^*) is a Lie bialgebroid if and only if $\mathcal{A} = T^*A$, with the structures just described, is a double Lie algebroid.*

PROOF. Assume that (A, A^*) is a Lie bialgebroid. The vertical structure on \mathcal{A} is the cotangent Lie algebroid structure for the Poisson structure on A . The anchor of this is a morphism of double vector bundles $\pi_A^\#: T^*A \rightarrow TA$ over $a_*: A^* \rightarrow TM$ and id_A , inducing $-a_*^*: T^*M \rightarrow A$ on the cores. Now the horizontal structure has the cotangent Lie algebroid structure for the Poisson structure on A^* , transported via $R = R_A: T^*A^* \rightarrow T^*A$. So the condition that $\pi_A^\#$ is a morphism of Lie algebroids over a_* with respect to the horizontal structure is precisely 5.1.

On the other hand, the anchor for the horizontal structure is

$$\pi_{A^*}^\# \circ R^{-1}: T^*A \rightarrow T(A^*),$$

and this is a morphism of double vector bundles over $a: A \rightarrow TM$ and id_{A^*} , inducing $+a^*$ on the cores. Since $R^{-1} = R_{A^*}$, the condition that this anchor be a morphism with respect

to the vertical structure is precisely the dual form of 5.1, to which 5.1 is equivalent by [26, 3.10] or [12].

The vertical dual of \mathcal{A} is the tangent double vector bundle as in Figure 1(b). Being the dual of a Lie algebroid, the vertical structure of this has a Poisson structure; this is the tangent lift of the Poisson structure on A [26, 5.6]. The corresponding Poisson tensor is

$$\pi_{TA}^\# = J_A \circ T(\pi_A^\#) \circ \theta_A^{-1}$$

where $J_A: T^2A \rightarrow T^2A$ is the canonical involution for the manifold A and $\theta_A: T(T^*A) \rightarrow T^*(TA)$ is the canonical map α of [34], denoted J' in [26, 5.4].

We must check that this Poisson structure coincides with that induced from the dual of \mathcal{A}^{*H} . Note that we have $K^* = TM$ here and to avoid confusion we drop the \dagger notation in this case and denote all duals over TM by \bullet . Duals over A^* will be denoted $^{*A^*}$.

Consider $I \circ R^{*A^*}: (T^*A)^{*A^*} \rightarrow T^\bullet A$. This preserves the core A^* and the side A^* but reverses the side TM . Define $W = -I \circ R^{*A^*}$ where the minus is for the bundle over A^* . Then $W^\bullet: TA \rightarrow (\mathcal{A}^{*H})^\bullet$ and the reader can check that $Z_V^{-1} = -W^\bullet$ where the heavy minus is over TM . See alternatively [22, 3.3].

To prove that $Z_V: (\mathcal{A}^{*V})^\bullet \rightarrow TA$ is an isomorphism of Lie algebroids over TM we must show that W is an anti-Poisson map. This may be done directly or by observing that $-W$ is, in terms of the double Lie algebroid $\mathcal{A}' = TA$ of 4.6, the map $(Z'_V)^\dagger = Z'_H$.

So we have $\mathcal{A}^{*V} = TA$ and $(\mathcal{A}^{*V})^\bullet = T^\bullet A$ and Condition III follows from the next result. We use I to replace $T^\bullet A$ by TA^* .

Lemma 5.3 *Given that (A, A^*) is a Lie bialgebroid on M , the tangent prolongation structures make (TA, TA^*) a Lie bialgebroid on TM with respect to the tangent pairing.*

PROOF. We use the bialgebroid criterion of 5.1. We must prove that

$$T^*(TA^*) \xrightarrow{\tilde{R}} T^*(TA) \xrightarrow{\pi_{TA}^\#} T^2A \quad (10)$$

is a morphism of Lie algebroids over the anchor of TA^* ; by [26, 5.1] this anchor is $J_M \circ T(a_*): TA^* \rightarrow T^2M$. Here \tilde{R} is the canonical map R for $TA \rightarrow TM$, transported using $I: TA^* \rightarrow T^\bullet A$. The domain of (10) is the cotangent Lie algebroid for the Poisson structure on TA^* , which Poisson structure—again by [26, 5.6]—is both the tangent lift of the Poisson structure on A^* and the dual (via I) of the prolongation Lie algebroid structure on TA . The target of (10) is the iterated tangent prolongation of the Lie algebroid structure of A .

Now $\tilde{R} = \theta_A \circ T(R_A) \circ \theta_{A^*}^{-1}$ and so

$$\pi_{TA}^\# \circ \tilde{R} = J_A \circ T(\pi_A^\# \circ R_A) \circ \theta_{A^*}^{-1}.$$

We know that $\pi_A^\# \circ R_A: T^*A^* \rightarrow TA$ is a morphism of Lie algebroids over a_* , so $T(\pi_A^\# \circ R_A)$ is a morphism of the prolongation structures over $T(a_*)$. We need two further observations.

Firstly, for any Poisson manifold, $\theta_P: T(T^*P) \rightarrow T^*(TP)$ is an isomorphism of Lie algebroids over TP from the tangent prolongation of the cotangent Lie algebroid structure on T^*P to the cotangent Lie algebroid of the tangent Poisson structure [24, 2.13]. We apply this to $P = A^*$.

Secondly, $J_A: T^2A \longrightarrow T^2A$ is a Lie algebroid automorphism over J_M of the iterated prolongation of the given Lie algebroid structure on A .

Putting these facts together, we have that $\pi_{TA}^\# \circ \widetilde{R}$ is a Lie algebroid morphism. ■

Now conversely suppose that A is a Lie algebroid on M and that A^* has a Lie algebroid structure, not *a priori* related to the structure on A . Consider $\mathcal{A} = T^*A$ with the two cotangent Lie algebroid structures arising from the Poisson structures on A^* and A , and suppose that these structures make \mathcal{A} a double Lie algebroid.

Then in particular the anchor of the horizontal structure must be a Lie algebroid morphism with respect to the other structures, as in Condition II, and this is

$$T^*A \xrightarrow{R_{A^*}} T^*A^* \xrightarrow{\pi_{A^*}^\#} TA^*$$

That this be a Lie algebroid morphism over $a: A \longrightarrow TM$ is precisely the dual form of 5.1.

This completes the proof of Theorem 5.2. ■

Recall the Manin triple characterization of a Lie bialgebra, as given in [18]: Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ the vector space direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra bracket defined in terms of the two coadjoint representations. This bracket is invariant under the pairing $\langle X + \varphi, Y + \psi \rangle = \psi(X) + \varphi(Y)$ and both \mathfrak{g} and \mathfrak{g}^* are coisotropic subalgebras. Conversely, if a Lie algebra \mathfrak{d} is a vector space direct sum $\mathfrak{g} \oplus \mathfrak{h}$, both of which are coisotropic with respect to an invariant pairing of \mathfrak{d} with itself, then $\mathfrak{h} \cong \mathfrak{g}^*$ and $(\mathfrak{g}, \mathfrak{h})$ is a Lie bialgebra, with \mathfrak{d} as the double.

Two aspects of this result concern us here. Firstly, it characterizes the notion of Lie bialgebra in terms of a single Lie algebra structure on \mathfrak{d} , the conditions being expressed in terms of the simple notion of pairing. Secondly, the roles of the two Lie algebras \mathfrak{g} and \mathfrak{g}^* are completely symmetric; it is an immediate consequence of the Manin triple result that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is so.

In considering a corresponding characterization for Lie bialgebroids, the most important difference to note is that the structure on the double is no longer over the same base as the given Lie algebroids. This is to be expected in view of the results of [20, §2] for the double groupoid case. There it is proved that if a double groupoid $(S; H, V; M)$ has trivial core (that is, the only elements of S to have two touching sides which are identity elements, are those which are identities for both structures), then there is a third groupoid structure on S , over base M , called in [20, p.200] the *diagonal structure* and denoted S_D . With respect to this structure on S , the identity maps from H and V into S are morphisms over M , and S as a manifold is $H \times_M V$. This diagonal structure is, in the case where H and V are dual Poisson groups, precisely the structure of the double group. The existence of the diagonal structure, in the general formulation given in [20, §2], depends crucially on the fact that S has trivial core; that is, that S is vacant. Since the core of the double vector bundle T^*A , for A a vector bundle on M , is T^*M , we expect T^*A to possess a Lie algebroid structure over M only when M is a point.

The role played in the bialgebra case by the Lie algebra structure of $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is taken, for Lie bialgebroids, by the two structures on T^*A (the bases of which are A and A^*). In place of a characterization in terms of a single Lie algebroid structure on T^*A with base M , Theorem 5.2 gives a characterization in terms of the two Lie algebroid structures on T^*A . The role of the pairing in the bialgebra case is taken in 5.2 by the isomorphism R .

There are considerable differences between the general notion of Courant algebroid and that of double Lie algebroid. Firstly, it seems clear that both concepts include examples not covered by the other. Open Question 1 of [15] asks for Courant algebroids which do not arise from Lie bialgebroids; it will be very surprising if these do not exist. There are certainly many classes of double Lie algebroids which are not the cotangent doubles of Lie bialgebroids, and it is not clear whether the correspondence between the Courant algebroid of a Lie bialgebroid and its cotangent double Lie algebroid can be extended to more general cases of either concept.

The notion of Courant algebroid provides a generalization to arbitrary bialgebroids of the Manin triple theorem and, perhaps most importantly, provides a setting for the study of Dirac structures. The notion of double Lie algebroid, on the other hand, while providing an alternative form of the Manin triple theorem for Lie bialgebroids, is also intended to give (as is argued in §4, §6, and the Introduction to [20]) a unification of iterated and second-order constructions in the foundations of differential geometry, and a general setting for duality phenomena. The two concepts are not only different, but differ in the nature of the problems which they are designed to address.

6 MATCHED PAIRS AND VACANT DOUBLE LIE ALGEBROIDS

The history of matched pairs of Lie algebras was briefly summarized in the Introduction. The corresponding concept of matched pair of Lie groups [18], [28] was extended to groupoids in [20]. In [29], Mokri differentiated the twisted automorphism equations of [20] to obtain conditions on a pair of Lie algebroid representations, of A on B and of B on A , which ensure that the direct sum vector bundle $A \oplus B$ has a Lie algebroid structure with A and B as subalgebroids. We quote the following.

Definition 6.1 [29, 4.2] *Let A and B be Lie algebroids on base M , with anchors a and b , and let $\rho: A \rightarrow \text{CDO}(B)$ and $\sigma: B \rightarrow \text{CDO}(A)$ be representations of A on the vector bundle B and of B on the vector bundle A . Then A and B together with ρ and σ form a matched pair if the following equations hold for all $X, X_1, X_2 \in \Gamma A$, $Y, Y_1, Y_2 \in \Gamma B$:*

$$\begin{aligned}\rho_X([Y_1, Y_2]) &= [\rho_X(Y_1), Y_2] + [Y_1, \rho_X(Y_2)] + \rho_{\sigma_{Y_2}(X)}(Y_1) - \rho_{\sigma_{Y_1}(X)}(Y_2), \\ \sigma_Y([X_1, X_2]) &= [\sigma_Y(X_1), X_2] + [X_1, \sigma_Y(X_2)] + \sigma_{\rho_{X_2}(Y)}(X_1) - \sigma_{\rho_{X_1}(Y)}(X_2), \\ a(\sigma_Y(X)) - b(\rho_X(Y)) &= [b(Y), a(X)].\end{aligned}$$

The notation $\text{CDO}(E)$ is defined in §3.

Proposition 6.2 [29, 4.3] *Given a matched pair, there is a Lie algebroid structure on the direct sum vector bundle $A \oplus B$, with anchor $c(X \oplus Y) = a(X) + b(Y)$ and bracket*

$$\begin{aligned}[X_1 \oplus Y_1, X_2 \oplus Y_2] &= \\ &= \{[X_1, X_2] + \sigma_{Y_1}(X_2) - \sigma_{Y_2}(X_1)\} \oplus \{[Y_1, Y_2] + \rho_{X_1}(Y_2) - \rho_{X_2}(Y_1)\}.\end{aligned}$$

Conversely, if $A \oplus B$ has a Lie algebroid structure making $A \oplus 0$ and $0 \oplus B$ Lie subalgebroids, then ρ and σ defined by $[X \oplus 0, 0 \oplus Y] = -\sigma_Y(X) \oplus \rho_X(Y)$ form a matched pair.

We now show that matched pairs correspond precisely to double Lie algebroids with zero core. The following definition is a natural sequel to [20, 2.11, 4.10].

Definition 6.3 *A double Lie algebroid $(\mathcal{A}; A^H, A^V; M)$ is vacant if the combination of the two projections, $(\tilde{q}_V, \tilde{q}_H): \mathcal{A} \longrightarrow A^H \times_M A^V$ is a diffeomorphism.*

Consider a vacant double Lie algebroid, which we will write here as $(\mathcal{A}; A, B; M)$. Note that $\mathcal{A} \longrightarrow A^H$ and $\mathcal{A} \longrightarrow A^V$ are the pullback bundles $q_A^! B$ and $q_B^! A$. The two duals are $\mathcal{A}^{*H} = A^* \oplus B$ and $\mathcal{A}^{*V} = A \oplus B^*$, as vector bundles over M , and the duality is (see [22, 3.4])

$$\langle X + \psi, \varphi + Y \rangle = \langle \varphi, X \rangle - \langle \psi, Y \rangle. \quad (11)$$

The horizontal bundle projection $\tilde{q}_A: \mathcal{A} \longrightarrow B$ is a morphism of Lie algebroids over $q_A: A \longrightarrow M$ and since it is a fibrewise surjection, it defines an action of B on q_A as in [8, §2]. Namely, each section Y of B induces the pullback section $1 \otimes Y$ of $q_A^! B$ and this induces a vector field $\eta(Y) = \tilde{b}(1 \otimes Y)$ on A , where $\tilde{b}: \mathcal{A} \longrightarrow TA$ is the anchor of the vertical structure. By Conditions I and II, $\eta(Y)$ is linear over the vector field $b(Y)$ on M , in the sense of [27, §1]; that is, $\eta(Y)$ is a vector bundle morphism $A \longrightarrow TA$ over $b(Y): M \longrightarrow TM$. It follows that $\eta(Y)$ defines covariant differential operators $\sigma_Y^{(*)}$ on A^* and σ_Y on A by

$$\eta(Y)(\ell_\varphi) = \ell_{\sigma_Y^{(*)}(\varphi)}, \quad \langle \varphi, \sigma_Y(X) \rangle = b(Y)\langle \varphi, X \rangle - \langle \sigma_Y^{(*)}(\varphi), X \rangle \quad (12)$$

where $\varphi \in \Gamma A^*$, $X \in \Gamma A$, and ℓ_φ denotes the function $A \longrightarrow \mathbb{R}$, $X \longmapsto \langle \varphi(q_A X), X \rangle$; see [27, §2]. Since \tilde{q}_A is a Lie algebroid morphism, it follows that σ is a representation of B on the vector bundle A .

Likewise, \tilde{q}_B is a morphism of Lie algebroids over q_B and for each $X \in \Gamma A$ we obtain a linear vector field $\xi(X) \in \mathcal{X}(B)$ over $a(X)$. We define covariant differential operators $\rho_X^{(*)}$ on B^* and ρ_X on B by

$$\xi(X)(\ell_\psi) = \ell_{\rho_X^{(*)}(\psi)}, \quad \langle \psi, \rho_X(Y) \rangle = a(X)\langle \psi, Y \rangle - \langle \rho_X^{(*)}(\psi), Y \rangle. \quad (13)$$

Again, $\rho_X^{(*)}$ and ρ_X are representations of A .

In fact (see [8, §2]) the two Lie algebroid structures on \mathcal{A} are action Lie algebroids determined by the actions $Y \longmapsto \eta(Y)$ and $X \longmapsto \xi(X)$. It follows that the dual Poisson structures are semi-direct in a general sense, but we will proceed on an ad hoc basis.

For a general vector bundle, the functions on the dual are generated by the linear functions and the pullbacks from the base manifold. In the case of a pullback bundle such as $q_B^! A$, one can refine this description a little further.

Since the two vector bundle structures on $A \times_M B$ are pullbacks the four classes of functions used in §3 simplify slightly. Namely, if $\pi_B: q_B^! A^* \longrightarrow A^*$ is $(\varphi, Y) \longmapsto \varphi$, and $\tilde{q}_{A^*}: q_B^! A^* \longrightarrow B$ is the bundle projection, then the functions on $q_B^! A^*$ are generated by all

$$\ell_X \circ \pi_B, \quad \ell_\psi \circ \tilde{q}_{A^*} \quad \text{and} \quad f \circ q_B \circ \tilde{q}_{A^*}$$

where $X \in \Gamma A$, $\psi \in \Gamma B^*$ and $f \in C(M)$. Now the Poisson structure on $q_B^! A^*$ is characterized by

$$\begin{aligned} \{\ell_{X_1} \circ \pi_B, \ell_{X_2} \circ \pi_B\} &= \ell_{[X_1, X_2]} \circ \pi_B, \\ \{\ell_X \circ \pi_B, \ell_\psi \circ \tilde{q}_{A^*}\} &= \ell_{\rho_X^{(*)}(\psi)} \circ \tilde{q}_{A^*}, \\ \{\ell_X \circ \pi_B, f \circ q_B \circ \tilde{q}_{A^*}\} &= a(X)(f) \circ q_B \circ \tilde{q}_{A^*}, \\ \{F_1 \circ \tilde{q}_{A^*}, F_2 \circ \tilde{q}_{A^*}\} &= 0, \end{aligned} \quad (14)$$

where F_1, F_2 are any smooth functions on B . Similarly,

$$\begin{aligned} \{\ell_{Y_1} \circ \pi_A, \ell_{Y_2} \circ \pi_A\} &= \ell_{[Y_1, Y_2]} \circ \pi_A, \\ \{\ell_Y \circ \pi_A, \ell_\varphi \circ \tilde{q}_{B^*}\} &= \ell_{\sigma_Y^{(*)}(\varphi)} \circ \tilde{q}_{B^*}, \\ \{\ell_Y \circ \pi_A, f \circ q_A \circ \tilde{q}_{B^*}\} &= b(Y)(f) \circ q_A \circ \tilde{q}_{B^*}, \\ \{G_1 \circ \tilde{q}_{B^*}, G_2 \circ \tilde{q}_{B^*}\} &= 0, \end{aligned} \quad (15)$$

Now these Poisson structures induce Lie algebroid structures on the direct sum bundles $A \oplus B^*$ and $A^* \oplus B$ over M . Consider first a section $X \oplus \psi$ of $\Gamma(A \oplus B^*)$. Via the pairing (11), this induces a linear function on $q_B^! A^*$, namely

$$\ell_{X \oplus \psi}^\dagger = \ell_X \circ \pi_B - \ell_\psi \circ \tilde{q}_{A^*}$$

where ℓ^\dagger refers to the pairing (11). By following through the equations (14) and (15) one obtains the following.

Lemma 6.4 *The Lie algebroid structure on $A \oplus B^*$ induced as above has anchor $e(X \oplus \psi) = a(X)$ and bracket*

$$[X_1 \oplus \psi_1, X_2 \oplus \psi_2] = [X_1, X_2] \oplus \{\rho_{X_1}^{(*)}(\psi_2) - \rho_{X_2}^{(*)}(\psi_1)\}. \quad (16)$$

The Lie algebroid structure on $A^ \oplus B$ induced as above has anchor $e_*(\varphi \oplus Y) = -b(Y)$ and bracket*

$$[\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2] = \{\sigma_{Y_2}^{(*)}(\varphi_1) - \sigma_{Y_1}^{(*)}(\varphi_2)\} \oplus [Y_2, Y_1]. \quad (17)$$

Thus $A \oplus B^*$ is the semi-direct product (over the base M , in the sense of [19]) of A with the vector bundle B^* with respect to $\rho^{(*)}$. However $A^* \oplus B$ is the opposite Lie algebroid to the semi-direct product of B with A^* .

We can now apply Condition III to $A \oplus B^*$ and $A^* \oplus B$. For brevity write $E = A \oplus B^*$. Recall [26], [12] that (E, E^*) is a Lie bialgebroid if and only if

$$d^E[\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2] = [d^E(\varphi_1 \oplus Y_1), \varphi_2 \oplus Y_2] + [\varphi_1 \oplus Y_1, d^E(\varphi_2 \oplus Y_2)] \quad (18)$$

for all $\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2 \in \Gamma E^*$. It follows [26, 3.4] that for any $f \in C(M)$, $X \oplus \psi \in \Gamma E$,

$$L_{d^E f}(X \oplus \psi) + [d^{E^*} f, X \oplus \psi] = 0. \quad (19)$$

It is easy to check that

$$d^E f = d^A f \oplus 0, \quad d^{E^*} f = 0 \oplus d^B f;$$

note that these imply that the Poisson structure induced on M by the Lie bialgebroid (E, E^*) is zero. Now the Lie derivative in (19) is a standard Lie derivative for E^* and so

$$\begin{aligned} \langle L_{d^E f}(X \oplus \psi), \varphi \oplus Y \rangle &= e_*(d^E f) \langle X \oplus \psi, \varphi \oplus Y \rangle - \langle X \oplus \psi, [d^E f, \varphi \oplus Y] \rangle \\ &= 0 - \langle X \oplus \psi, \sigma_Y^{(*)}(d^A f) \oplus 0 \rangle \\ &= \langle d^A f, \sigma_Y(X) \rangle - b(Y) \langle d^A f, X \rangle \\ &= a(\sigma_Y(X))(f) - b(Y)a(X)(f) \end{aligned} \quad (20)$$

where we used (12) and (17). Expanding out the bracket term in (19) in a similar way, we obtain the third equation in 6.1.

Now consider the bialgebroid equation (18). We calculate this in the case $\varphi_1 = \varphi_2 = 0$, with arguments $0 \oplus \psi_1, X_2 \oplus 0$. With these values we refer to (18) as equation (18)₀. First we need the following lemma, which is a straightforward calculation.

Lemma 6.5

$$d^E(\varphi \oplus Y)(X_1 \oplus \psi_1, X_2 \oplus \psi_2) = (d^A \varphi)(X_1, X_2) + \langle \psi_1, \rho_{X_2}(Y) \rangle - \langle \psi_2, \rho_{X_1}(Y) \rangle.$$

The LHS of (18)₀ is easily seen to be $\langle \psi_1, \rho_{X_2}[Y_2, Y_1] \rangle$. On the RHS, consider the second term first. Regarding the bracket as a Lie derivative, we have

$$\begin{aligned} \langle L_{0 \oplus Y_1}(d^E(0 \oplus Y_2)), (0 \oplus \psi_1) \wedge (X_2 \oplus 0) \rangle &= L_{0 \oplus Y_1} \langle d^E(0 \oplus Y_2), (0 \oplus \psi_1) \wedge (X_2 \oplus 0) \rangle \\ &\quad - \langle d^E(0 \oplus Y_2), L_{0 \oplus Y_1}((0 \oplus \psi_1) \wedge (X_2 \oplus 0)) \rangle \end{aligned} \quad (21)$$

Since $e_*(0 \oplus Y_1) = -b(Y_1)$, the first term is $-b(Y_1) \langle \psi_1, \rho_{X_2}(Y_2) \rangle$. For the second term we need the following lemma.

Lemma 6.6 *For any $\varphi \oplus Y \in \Gamma E^*$ and $X \oplus \psi \in \Gamma E$, we have $L_{\varphi \oplus Y}(X \oplus \psi) = -\sigma_Y(X) \oplus \overline{\psi}$ where for any $Y' \in B$,*

$$\langle \overline{\psi}, Y' \rangle = -b(Y) \langle \psi, Y' \rangle + \langle \sigma_{Y'}^{(*)}(\varphi), X \rangle + \langle \psi, [Y, Y'] \rangle.$$

PROOF. This is a Lie derivative for E^* and applying the same device as in (20) we have, for any $\varphi \oplus Y' \in \Gamma E^*$,

$$\langle \varphi' \oplus Y', L_{\varphi \oplus Y}(X \oplus \psi) \rangle = -\langle \varphi', \sigma_Y(X) \rangle + b(Y) \langle \psi, Y' \rangle - \langle \sigma_{Y'}^{(*)}(\varphi), X \rangle + \langle \psi, [Y', Y] \rangle.$$

Setting $Y' = 0$ and $\varphi' = 0$ in turn gives the result. ■

Now expand out the Lie derivative of the wedge product in (21) and apply Lemma 6.6. One obtains for the second term on the RHS of (18)₀

$$-\langle \psi_1, [Y_1, \rho_{X_2}(Y_2)] \rangle + \langle \psi_1, \rho_{\sigma_{Y_1}(X_2)}(Y_2) \rangle.$$

The first term on the RHS of (18)₀ is easily obtained from this, and combining with the LHS we have the first equation in 6.1. The second equation is obtained in a similar way from the dual form of (18).

This completes the proof of the first part of the following result.

Theorem 6.7 *Let $(\mathcal{A}; A, B; M)$ be a vacant double Lie algebroid. Then the two Lie algebroid structures on \mathcal{A} are action Lie algebroids corresponding to actions which define representations ρ , of A on B , and σ , of B on A , with respect to which A and B form a matched pair.*

Conversely, let A and B be a matched pair of Lie algebroids with respect to ρ and σ . Then the action of A on q_B induced by ρ and the action of B on q_A induced by σ define Lie algebroid structures on $\mathcal{A} = A \times_M B$ with respect to which $(\mathcal{A}; A, B; M)$ is a vacant double Lie algebroid.

PROOF. It remains to prove the second statement. We first verify the bialgebroid condition (18).

For $(18)_0$ it suffices to retrace the steps of the direct argument. Now (18) with $\varphi_1 = \varphi_2 = 0$ and any $(X_1 + \psi_1, X_2 + \psi_2) = (X_1 + 0, 0 + \psi_2) + (0 + \psi_1, X_2 + 0)$ follows by skew-symmetry of all terms in (18).

For (18) with $Y_1 = Y_2 = 0$ and any arguments, it is straightforward to verify that all terms are identically zero.

Consider the case $\varphi_1 = 0$, $Y_2 = 0$, with arbitrary arguments. Calculating by the same methods as in the proof of the direct statement, and expressing all contragredient expressions in terms of ρ and σ , we see that we must prove that

$$\begin{aligned} & -a(X_1)b(Y_1)\langle\varphi_2, X_2\rangle + a(X_2)b(Y_1)\langle\varphi_2, X_1\rangle + a(X_1)\langle\varphi_2, \sigma_{Y_1}(X_2)\rangle \\ & \quad - a(X_2)\langle\varphi_2, \sigma_{Y_1}(X_1)\rangle + b(Y_1)\langle\varphi_2, [X_1, X_2]\rangle - \langle\varphi_2, \sigma_{Y_1}[X_1, X_2]\rangle \\ & = -b(Y_1)a(X_1)\langle\varphi_2, X_2\rangle + b(Y_1)a(X_2)\langle\varphi_2, X_1\rangle + b(Y_1)\langle\varphi_2, [X_1, X_2]\rangle \\ & \quad + a(\sigma_{Y_1}(X_1))\langle\varphi_2, X_2\rangle - a(X_2)\langle\varphi_2, \sigma_{Y_1}(X_1)\rangle - \langle\varphi_2, [\sigma_{Y_1}(X_1), X_2]\rangle \\ & \quad + a(X_1)\langle\varphi_2, \sigma_{Y_1}(X_2)\rangle - a(\sigma_{Y_1}(X_2))\langle\varphi_2, X_1\rangle - \langle\varphi_2, [X_1, \sigma_{Y_1}(X_2)]\rangle \\ & \quad + b(\rho_{X_2}(Y_1))\langle\varphi_2, X_1\rangle - \langle\varphi_2, \sigma_{\rho_{X_2}(Y_1)}(X_1)\rangle - b(\rho_{X_1}(Y_1))\langle\varphi_2, X_2\rangle \\ & \quad + \langle\varphi_2, \sigma_{\rho_{X_1}(Y_1)}(X_2)\rangle. \end{aligned}$$

Here six terms cancel in pairs, a further five on account of the second equation in 6.1, and the remaining eight by a double application of the last equation in 6.1. This completes the proof that $(A \ltimes B^*, \overline{A^* \ltimes B})$ is a Lie bialgebroid.

The remaining parts of Condition III, and Condition I, are straightforward. We verify Condition II. We need the following result.

Proposition 6.8 [27, 2.5] *Let (ξ, x) be a linear vector field on a vector bundle $E \longrightarrow M$, and let D be the corresponding element of $\Gamma \text{CDO}(E)$. Then for all $\mu \in \Gamma E, m \in M$,*

$$\xi(\mu(m)) = T(\mu)(x(m)) - D(\mu)^\dagger(\mu(m)). \quad (22)$$

In particular for any vector fields x, y on M , and the complete lift $\tilde{x} = J \circ T(x) \in \mathcal{X}(TM)$, where $J: T^2M \longrightarrow T^2M$ is the canonical involution,

$$T(y)(x(m)) = \tilde{x}(y(m)) + [x, y]^\dagger(y(m)). \quad (23)$$

Because $T(y)(x(m))$ and $\tilde{x}(y(m))$, as elements of T^2M , both project to $x(m)$ under $T(p)$ and to $y(m)$ under p_{TM} , it follows from (23) and the interchange law for T^2M that

$$T(y)(x(m)) = \tilde{x}(y(m)) \mathrel{++} [x, y]^\wedge(x(m)), \quad (24)$$

where for any vector bundle $E \rightarrow M$ and $\mu \in \Gamma E$, $\widehat{\mu}$ denotes the section of $TE \rightarrow TM$ with $\widehat{\mu}(z) = T(0)(z) + \overline{\mu(m)}$ for $z \in T_m M$; see [27, §2].

Returning to the proof of Theorem 6.7, we must show that the anchor $\tilde{a}: q_A \rtimes B \rightarrow TB$ is a Lie algebroid morphism over $a: A \rightarrow TM$, where the domain is the action Lie algebroid for the action of B on q_A , and the target is the prolongation of B . First we verify that \tilde{a} commutes with the anchor \tilde{b} of the domain and the anchor $J \circ T(b)$ of the target. Applying (22) to (13) we obtain, for any $X \in \Gamma A$, $Y \in \Gamma B$,

$$\tilde{a}(X(m), Y(m)) = \xi(X)(Y(m)) = T(Y)(aX(m)) - \rho_X(Y)^\dagger(Y(m)). \quad (25)$$

Then applying $T(b)$ to both sides and writing $x = aX, y = bY$, we have

$$(T(b) \circ \tilde{a})(X(m), Y(m)) = T(y)(x(m)) - (b(\rho_X(Y)))^\dagger(y(m))$$

where we used the fact that $T(b)$ is a morphism of double vector bundles $TB \rightarrow T^2M$ with core b .

Using the third equation of 6.1, and (23), this becomes

$$\tilde{x}(y(m)) - (a(\sigma_Y(X)))^\dagger(y(m)) = \tilde{x}(y(m)) \mathrel{-} (a(\sigma_Y(X)))^\wedge(x(m)),$$

where $\mathrel{-}$ is the prolonged subtraction in $T(p): T^2M \rightarrow TM$. Applying J to this, then applying (22) to (12) and proceeding as before, gives

$$T(x)(y(m)) - (a(\sigma_Y(X)))^\dagger(x(m)) = T(a)(\eta(Y)(X(m))) = T(a)(\tilde{b}(X(m), Y(m))).$$

Thus $T(a) \circ \tilde{b} = J \circ T(b) \circ \tilde{a}$ as required.

We now verify the bracket condition, as given in [8]. Take $Y \in \Gamma B$ and denote by \overline{Y} the pullback section of $q_A \rtimes B \rightarrow A$. (It suffices to consider such sections, since they generate $\Gamma_A(q_A \rtimes B)$.) In general there is no section of $TB \rightarrow TM$ to which \overline{Y} projects under \tilde{a} . However from (25) it follows that

$$\tilde{a} \circ \overline{Y} = 1 \otimes T(Y) \mathrel{-} R_Y$$

where $1 \otimes T(Y)$ is the pullback of $T(Y) \in \Gamma_{TM} TB$ across a , and $R_Y \in \Gamma(a^! TB)$ is defined by

$$R_Y(X) = (X, \rho_X(Y))^\wedge(aX).$$

As with any section of a pullback, R_Y has a tensor decomposition which here can be taken to be of the form

$$\sum \ell_{\varphi_i} \otimes \widehat{W}_i \mathrel{++} \sum (f_j \circ q) \otimes \widehat{V}_j$$

where $\varphi_i \in \Gamma A^*$, $f_j \in C(M)$, $W_i, V_j \in \Gamma B$ and ℓ_φ denotes the linear function $A \rightarrow \mathbb{R}$, $X \mapsto \langle \varphi(qX), X \rangle$. (The addition $\mathrel{++}$ is the prolonged addition in $TB \rightarrow TM$.) Now $(f \circ q) \otimes \widehat{V} = (f \circ p \circ a) \otimes \widehat{V} = 1 \otimes (f \circ p) \cdot \widehat{V} = 1 \otimes \widehat{(fV)}$, where $p: TM \rightarrow M$ is the projection. The second group of terms can therefore be collapsed to a single $1 \otimes \widehat{V}$. Since

$R(0_m) = (0, 0)$ and $\ell_\varphi(0_m) = 0$ for all $0_m \in A$, it follows that $V = 0$ and we can take, for $Y_1, Y_2 \in \Gamma B$,

$$R_{Y_1} = \sum \ell_{\varphi_i} \otimes \widehat{W}_i, \quad R_{Y_2} = \sum \ell_{\psi_j} \otimes \widehat{Z}_j. \quad (26)$$

Following [8, 1.3], we must prove that $\tilde{a} \circ \overline{[Y_1, Y_2]}$ is given by

$$\begin{aligned} 1 \otimes [T(Y_1), T(Y_2)] &= \sum \ell_{\varphi_i} \otimes [\widehat{W}_i, T(Y_2)] - \sum \ell_{\psi_j} \otimes [T(Y_1), \widehat{Z}_j] \\ &\quad ++ \sum \ell_{\varphi_i} \ell_{\psi_j} \otimes [\widehat{W}_i, \widehat{Z}_j] - \sum \tilde{b}(\overline{Y_1})(\ell_{\psi_j}) \otimes \widehat{Z}_j ++ \sum \tilde{b}(\overline{Y_2})(\ell_{\varphi_i}) \otimes \widehat{W}_i \\ &= 1 \otimes T([Y_1, Y_2]) ++ \sum \ell_{\varphi_i} \otimes [Y_2, W_i]^\wedge - \sum \ell_{\psi_j} \otimes [Y_1, Z_j]^\wedge \\ &\quad - \sum \eta(Y_1)(\ell_{\psi_j}) \otimes \widehat{Z}_j ++ \sum \eta(Y_2)(\ell_{\varphi_i}) \otimes \widehat{W}_i \end{aligned}$$

where we used the definition of η in terms of the anchor \tilde{b} and the following relations, from [26, (27)], for the bracket in $TB \rightarrow TM$:

$$[T(Y_1), T(Y_2)] = T([Y_1, Y_2]), \quad [T(Y_1), \widehat{Y_2}] = [Y_1, Y_2]^\wedge, \quad [\widehat{Y_1}, \widehat{Y_2}] = 0,$$

where $Y_1, Y_2 \in \Gamma B$. Now equations (26) are equivalent to

$$\rho_X(Y_1) = \sum \langle \varphi_i, X \rangle W_i, \quad \rho_X(Y_2) = \sum \langle \psi_j, X \rangle Z_j,$$

where $X \in \Gamma A$, $Y_1, Y_2 \in \Gamma B$, so it follows that

$$\begin{aligned} [\rho_X(Y_1), Y_2] &= \sum \langle \varphi_i, X \rangle [W_i, Y_2] - \sum b(Y_2) \langle \varphi_i, X \rangle W_i, \\ \rho_{\sigma_{Y_2}(X)}(Y_1) &= \sum \langle \varphi_i, \sigma_{Y_2}(X) \rangle W_i, \end{aligned}$$

together with two similar equations. Using the first equation of 6.1 this leads to

$$\begin{aligned} \rho_X([Y_1, Y_2]) &= - \sum \langle \varphi_i, X \rangle [Y_2, W_i] - \sum b(Y_2) \langle \varphi_i, X \rangle W_i + \sum \langle \psi_j, X \rangle [Y_1, Z_j] \\ &\quad + \sum b(Y_1) \langle \psi_j, X \rangle Z_j + \sum \langle \varphi_i, \sigma_{Y_2}(X) \rangle W_i - \sum \langle \psi_j, \sigma_{Y_1}(X) \rangle Z_j, \end{aligned}$$

and by (12) this is

$$- \sum \langle \varphi_i, X \rangle [Y_2, W_i] + \sum \langle \psi_j, X \rangle [Y_1, Z_j] - \sum (\eta(Y_2)(\ell_{\varphi_i}) \circ X) W_i + \sum (\eta(Y_1)(\ell_{\psi_j}) \circ X) Z_j.$$

So $\tilde{a} \circ \overline{[Y_1, Y_2]} = 1 \otimes T([Y_1, Y_2]) - R_{[Y_1, Y_2]}$ indeed coincides with the RHS of (6), and brackets are preserved.

This completes the proof of Theorem 6.7. \blacksquare

In the process we have also proved the following result.

Theorem 6.9 *Let A and B be Lie algebroids on the same base M , and let $\rho: A \rightarrow \text{CDO}(B_0)$ and $\sigma: B \rightarrow \text{CDO}(A_0)$ be representations, where A_0 and B_0 denote the underlying vector bundles.*

Then A and B form a matched pair with respect to ρ and σ if and only if $(A \ltimes B^, \overline{A^* \rtimes B})$, with the Lie algebroid structures described in 6.4, form a Lie bialgebroid.*

Thus the three matched pair equations in 6.1 are embodied in the bialgebroid equation (18).

Theorem 6.9 in the Lie algebra case has been found very recently by Stachura [33], and in the setting of Lie–Rinehart algebras by Huebschmann [9].

In the case of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the Lie bialgebroid associated to the vacant double Lie algebroid is $(\mathfrak{g} \oplus \mathfrak{g}_0, \mathfrak{g}^* \oplus \mathfrak{g}_0^*)$ where the subscripts denote the abelianizations. This is of course consistent with 5.2 in the bialgebra case—which is both a bialgebroid and a matched pair.

One other example which should be mentioned briefly is that of affinoids. An *affinoid* [37] may be regarded as a vacant double Lie groupoid in which both side groupoids H and V are the graphs of simple foliations defined by surjective submersions $\pi_1: M \rightarrow Q_1$ and $\pi_2: M \rightarrow Q_2$. The corresponding double Lie algebroid was calculated in [23] to be a pair of conjugate flat partial connections adapted to the two foliations. The bialgebroid in this case is $(T^{\pi_1} M \oplus (T^{\pi_2} M)^*, (T^{\pi_1} M)^* \oplus T^{\pi_2} M)$ with semi-direct structures defined by the connections.

Theorem 6.7 actually shows the equivalence of three formulations: a vacant double Lie algebroid structure on $(\mathcal{A}; A, B; M)$ is equivalent to a matched pair structure on (A, B) , which in turn is equivalent to a Lie bialgebroid structure on $(A \ltimes B^*; A^* \rtimes B)$ of the type in 6.4. This provides confirmation, we believe, of the correctness of the definition of double Lie algebroid. It also provides a diagrammatic characterization of matched pairs of Lie algebroids, directly comparable to the diagrammatic description of matched pairs of group(oid)s given in [20, §2]. In the groupoid case, the twisted multiplicativity equations are fairly unintuitive, and we believe that the derivation of them directly from the vacant double groupoid axioms has been a significant clarification. In the Lie algebroid case the equations in 6.1 are again not simple and, unlike the groupoid case, are defined in terms of sections rather than elements. Nonetheless the characterization given by 6.7 is purely diagrammatic: recall that the characterization 5.1 of a Lie bialgebroid is formulated entirely in terms of the Poisson tensor and the canonical isomorphism R . Thus we have a definition of matched pair which can be formulated more generally in a category possessing pullbacks and suitable additive structure.

References

- [1] A. L. Besse. *Manifolds all of whose geodesics are closed*, volume 93 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1978.
- [2] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.
- [3] A. Coste, P. Dazord, and A. Weinstein. Groupoïdes symplectiques. In *Publications du Département de Mathématiques de l’Université de Lyon, I*, number 2/A-1987, pages 1–65, 1987.
- [4] V. G. Drinfel’d. Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation. *Soviet. Math. Dokl.*, 27:68–71, 1983.
- [5] V. G. Drinfel’d. Quantum groups. In A. M. Gleason, editor, *Proceedings of the International Congress of Mathematicians, Berkeley, 1986*, pages 798–820. American Mathematical Society, Providence, R. I., 1987.

- [6] V. G. Drinfel'd. On Poisson homogeneous spaces of Poisson–Lie groups. *Theo. Math. Phys.*, 95(2):226–227, 1993.
- [7] Pavel Etingof and Alexander Varchenko. Geometry and classification of solutions of the classical dynamical Yang-Baxter equation. *Comm. Math. Phys.*, 192(1):77–120, 1998.
- [8] P. J. Higgins and K. C. H. Mackenzie. Algebraic constructions in the category of Lie algebroids. *J. Algebra*, 129:194–230, 1990.
- [9] J. Huebschmann. Twilled Lie-Rinehart algebras and differential Batalin-Vilkovisky algebras. Preprint [math.dg/9811069](#), 1998.
- [10] M. V. Karasev. The Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fibre bundles of symplectic manifolds. I, II. *Selecta Math. Soviet.*, 8:213–234, 235–258, 1989. Preprint, Moscov. Inst. Electron Mashinostroeniya, 1981, deposited at VINITI, 1982.
- [11] Katarzyna Konieczna and Paweł Urbański. Double vector bundles and duality. *Arch. Math. (Brno)*, 35(1):59–95, 1999.
- [12] Y. Kosmann-Schwarzbach. Exact Gerstenhaber algebras and Lie bialgebroids. *Acta Appl. Math.*, 41:153–165, 1995.
- [13] Y. Kosmann-Schwarzbach and F. Magri. Poisson–Lie groups and complete integrability, I. Drinfeld bigebras, dual extensions and their canonical representations. *Ann. Inst. H. Poincaré Phys. Théor.*, 49:433–460, 1988.
- [14] Y. Kosmann-Schwarzbach and F. Magri. Poisson-Nijenhuis structures. *Ann. Inst. H. Poincaré Phys. Théor.*, 53:35–81, 1990.
- [15] Zhang-Ju Liu, Alan Weinstein, and Ping Xu. Manin triples for Lie bialgebroids. *J. Differential Geom.*, 45:547–574, 1997.
- [16] Jiang-Hua Lu. Poisson homogeneous spaces and Lie algebroids associated to Poisson actions. *Duke Math. J.*, 86:261–304, 1997.
- [17] Jiang-Hua Lu and A. Weinstein. Groupoïdes symplectiques doubles des groupes de Lie-Poisson. *C. R. Acad. Sci. Paris Sér. I Math.*, 309:951–954, 1989.
- [18] Jiang-Hua Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Differential Geom.*, 31:501–526, 1990.
- [19] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*. London Mathematical Society Lecture Note Series, no. 124. Cambridge University Press, 1987.
- [20] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry, I. *Adv. Math.*, 94(2):180–239, 1992.
- [21] K. C. H. Mackenzie. Drinfel'd doubles and Ehresmann doubles for Lie algebroids and Lie bialgebroids. *Electron. Res. Announc. Amer. Math. Soc.*, 4:74–87, 1998.

- [22] K. C. H. Mackenzie. On symplectic double groupoids and the duality of Poisson groupoids. *Internat. J. Math.*, 10:435–456, 1999.
- [23] K. C. H. Mackenzie. Affinoid structures and connections. In J. Grabowski and P. Urbański, editors, *Poisson geometry: Stanisław Zakrzewski In Memoriam*, number 51 in Banach Center Publications, pages 175–186, Warsaw, 2000.
- [24] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry, II. *Adv. Math.*, 154:46–75, 2000. 27pp.
- [25] K. C. H. Mackenzie. A unified approach to Poisson reduction. *Lett. Math. Phys. (to appear)*, 2000.
- [26] K. C. H. Mackenzie and Ping Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.*, 73(2):415–452, 1994.
- [27] K. C. H. Mackenzie and Ping Xu. Classical lifting processes and multiplicative vector fields. *Quarterly J. Math. Oxford (2)*, 49:59–85, 1998.
- [28] S. Majid. Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations. *Pacific J. Math.*, 141:311–332, 1990.
- [29] T. Mokri. Matched pairs of Lie algebroids. *Glasgow Math. J.*, 39:167–181, 1997.
- [30] J. Pradines. Fibrés vectoriels doubles et calcul des jets non holonomes. Notes polycopiées, Amiens, 1974.
- [31] J. Pradines. Remarque sur le groupoïde cotangent de Weinstein-Dazord. *C. R. Acad. Sci. Paris Sér. I Math.*, 306:557–560, 1988.
- [32] D. Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. Ph.d. thesis, University of California, Berkeley, 1999. math.DG/9910078.
- [33] P. Stachura. Double Lie algebras and Manin triples. Preprint, 1997. q-alg/9712040.
- [34] W. M. Tulczyjew. *Geometric formulation of physical theories*, volume 11 of *Monographs and Textbooks in Physical Science*. Bibliopolis, Naples, 1989.
- [35] A. Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16:101–104, 1987.
- [36] A. Weinstein. Coisotropic calculus and Poisson groupoids. *J. Math. Soc. Japan*, 40:705–727, 1988.
- [37] A. Weinstein. Affine Poisson structures. *Internat. J. Math.*, 1:343–360, 1990.
- [38] Ping Xu. On Poisson groupoids. *Internat. J. Math.*, 6(1):101–124, 1995.
- [39] Ping Xu. Gerstenhaber algebras and BV-algebras in Poisson geometry. *Comm. Math. Phys.*, 200(3):545–560, 1999.